

On the Energy Subcritical, Non-linear Wave Equation with Radial Data for $p \in (3, 5)$

Ruipeng Shen

August 13, 2012

1 Introduction

In this paper we will consider the energy subcritical, non-linear wave equation in \mathbb{R}^3 with radial initial data.

$$\begin{cases} \partial_t^2 u - \Delta u = \pm |u|^{p-1}u, & (x, t) \in \mathbb{R}^3 \times \mathbb{R}; \\ u|_{t=0} = u_0 \in \dot{H}^{s_p}(\mathbb{R}^3); \\ \partial_t u|_{t=0} = u_1 \in \dot{H}^{s_p-1}(\mathbb{R}^3). \end{cases} \quad (1)$$

Here $3 < p < 5$ and

$$s_p = \frac{3}{2} - \frac{2}{p-1}.$$

The positive sign in the non-linear term gives us the focusing case, while the negative sign indicates the defocusing case. The following quantity is called the energy of the solution. The energy is a constant in the whole lifespan of the solution, as long as it is well-defined.

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^3} (|\partial_t u(x, t)|^2 + |\nabla u(x, t)|^2) dx \mp \frac{1}{p+1} \int_{\mathbb{R}^3} |u(x, t)|^{p+1} dx. \quad (2)$$

Please note that the energy could be a negative number in the focusing case.

Previous Results in the Energy-critical Case In the energy-critical case, namely $p = 5$, the initial data is in the energy space $\dot{H}^1 \times L^2$. This automatically guarantees the existence of the energy by the Sobolev embedding. This kind of wave equations have been extensively studied. In the defocusing case, M. Grillakis (See [7, 8]) proved the global existence and scattering of the solution with any $\dot{H}^1 \times L^2$ initial data in 1990's. In the focusing case, however, the behavior of solutions is much more complicated. The solutions may scatter, blow up in finite time or even be independent of time. Please see [4, 9] for more details. In particular, a solution independent of time is usually called a ground state or a soliton. This kind of solutions are actually the solutions of the elliptic equation $-\Delta W(x) = |W(x)|^{p-1}W(x)$. We can write down all the nontrivial radial solitons explicitly as below. The letter λ here is an arbitrary positive parameter.

$$W(x) = \pm \frac{1}{\lambda^{1/2}} \left(1 + \frac{|x|^2}{3\lambda^2} \right)^{-1/2}. \quad (3)$$

Energy Subcritical Case We will consider the case $3 < p < 5$ in this paper, thus $1/2 < s_p < 1$. In this case the problem is critical in the space $\dot{H}^{s_p}(\mathbb{R}^3) \times \dot{H}^{s_p-1}(\mathbb{R}^3)$, because if $u(x, t)$ is a solution of (1) with initial data (u_0, u_1) , then for any $\lambda > 0$, the function

$$\frac{1}{\lambda^{3/2-s_p}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right)$$

is another solution of the equation (1) with the initial data

$$\left(\frac{1}{\lambda^{3/2-s_p}} u_0\left(\frac{x}{\lambda}\right), \frac{1}{\lambda^{5/2-s_p}} u_1\left(\frac{x}{\lambda}\right)\right),$$

which shares the same $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$ norm as the original initial data (u_0, u_1) . These scalings play an important role in our discussion of this problem.

Theorem 1.1. (Main Theorem) *Let u be a solution of the non-linear wave equation (1) with radial initial data $(u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$ and a maximal lifespan I so that*

$$\sup_{t \in I} \|(u, \partial_t u)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} < \infty. \quad (4)$$

Then u is global in time ($I = \mathbb{R}$) and scatters, i.e.

$$\|u(x, t)\|_{S(\mathbb{R})} < \infty, \text{ or equivalently } \|u(x, t)\|_{Y_{s_p}(\mathbb{R})} < \infty.$$

This is actually equivalent to saying that there exist two pairs (u_0^+, u_1^+) and (u_0^-, u_1^-) in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$ such that

$$\lim_{t \rightarrow \pm\infty} \|(u(t) - S(t)(u_0^\pm, u_1^\pm), \partial_t u(t) - \partial_t S(t)(u_0^\pm, u_1^\pm))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} = 0.$$

Here $S(t)(u_0^\pm, u_1^\pm)$ is the solution of the Linear Wave Equation with the initial data (u_0^\pm, u_1^\pm) .

Please refer to section 2 for the definition of the S and Y_s norms.

Remark on the Defocusing Case As in the energy-critical case, we expect that the solutions always scatter. Besides the radial condition, the main theorem depends on the assumption (4), which is expected to be true for all solutions. Unfortunately, as far as the author knows, no one actually knows how to prove it without additional assumptions.

Remark on the Focusing Case In the focusing case, the solutions may blow up in finite time. (Please see theorem 6.2, for instance) Thus the assumption (4) is a meaningful and essential condition rather than a technical one. The main theorem gives us the following rough classification of the radial solutions.

Proposition 1.2. *Let $u(t)$ be a solution of (1) in the focusing case with a maximal lifespan I and radial initial data $(u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$. Then one of the following holds for $u(x, t)$.*

- (I) (Blow-up) *The $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$ norm of $(u(t), \partial_t u(t))$ blows up, namely*

$$\sup_{t \in I} \|(u(t), \partial_t u(t))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} = +\infty.$$

- (II) (Scattering) *If the upper bound of the $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$ norm above is finite instead, namely, the assumption (4) holds, then $u(t)$ is a global solution (i.e $I = \mathbb{R}$) and scatters.*

Main Idea in this Paper The main idea to establish theorem 1.1 is to use the compactness/rigidity argument, namely to show

- (I) If the main theorem failed, it would break down at a minimal blow-up solution, which is almost periodic modulo scalings.
- (II) The minimal blow-up solution is in the energy space.
- (III) The minimal blow-up solution described above does not exist.

Step (I) The method of profile decomposition used here has been a standard way to deal with both the wave equation and the Schrödinger equation. Thus we will only give important statements instead of showing all the details. The other steps, however, depend on the specific problems. One could refer to [1] in order to understand what is the profile decomposition, and to [9, 12] in order to see why the profile decomposition leads to the existence of a minimal blow-up solution.

Step (II) We will combine the method used in my old paper [20] and a method used in C.E.Kenig and F.Merle's paper [11] on the supercritical case of the non-linear wave equation in \mathbb{R}^3 . The idea is to use the following fact. Given a radial solution $u(x, t)$ of the equation

$$\partial_t^2 u(x, t) - \Delta u(x, t) = F(x, t)$$

in the time interval I , if we define two functions $w, h : \mathbb{R}^+ \times I \rightarrow \mathbb{R}$, such that $w(|x|, t) = |x|u(x, t)$ and $h(|x|, t) = |x|F(x, t)$, then $w(r, t)$ is a solution of the one-dimensional wave equation $\partial_t^2 w(r, t) - \partial_r^2 w(r, t) = h(r, t)$. This makes it convenient to consider the integral

$$\int_{r_0 \pm t}^{4r_0 \pm t} |\partial_t w(r, t_0 + t) \mp \partial_r w(r, t_0 + t)|^2 dr.$$

as the parameter t moves.

Step (III) Given an energy estimate, all minimal blow-up solutions are not difficult to kill except for the soliton-like solutions in the focusing case. As I mentioned earlier, this kind of solutions actually exist in the energy-critical case. The ground states given in (3) are perfect examples. In the energy subcritical case, however, the soliton does not exist at all. More precisely, none of the solutions of the corresponding elliptic equation is in the right space \dot{H}^{s_p} . This fact enables us to gain a contradiction by showing a soliton-like minimal blow-up solution must be a real soliton, which does not exist, using a new method introduced by Thomas Duyckaerts, Carlos Kenig and Frank Merle. They classified all radial solutions of the energy-critical, focusing wave equation in their recent paper [4] using this "channel of energy" method.

Remark on the Supercritical Case Simultaneously to this work, Thomas Duyckaerts, Carlos Kenig and Frank Merle [3] proved that similar results to those in this paper also hold in the supercritical case $p > 5$ of the focusing wave equation, using the compactness/rigidity argument, a point-wise estimate on "compact" solutions obtained in the paper [11] and the "channel of energy" method mentioned above.

2 Preliminary results

2.1 Local Theory with $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$ Initial Data

In this section, we will review the theory for the Cauchy problem of the nonlinear wave equation (1) with initial data in the critical space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$. The same local theory works in both the focusing and defocusing cases. It could be also applied to the non-radial case.

Space-time Norm Let I be an interval of time. The space-time norm is defined by

$$\|v(x, t)\|_{L^q L^r(I \times \mathbb{R}^3)} = \left(\int_I \left(\int_{\mathbb{R}^3} |v(x, t)|^r dx \right)^{q/r} dt \right)^{1/q}.$$

This is used in the following Strichartz estimates.

Proposition 2.1. Generalized Strichartz Inequalities *(Please see proposition 3.1 of [6], here we use the Sobolev version in \mathbb{R}^3)* Let $2 \leq q_1, q_2 \leq \infty$, $2 \leq r_1, r_2 < \infty$ and $\rho_1, \rho_2, s \in \mathbb{R}$ with

$$1/q_i + 1/r_i \leq 1/2; \quad i = 1, 2.$$

$$1/q_1 + 3/r_1 = 3/2 - s + \rho_1.$$

$$1/q_2 + 3/r_2 = 1/2 + s + \rho_2.$$

In particular, if $(q_1, r_1, s, \rho_1) = (q, r, m, 0)$ satisfies the conditions above, we say (q, r) is an m -admissible pair.

Let u be the solution of the following linear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = F(x, t), & (x, t) \in \mathbb{R}^3 \times \mathbb{R}; \\ u|_{t=0} = u_0 \in \dot{H}^s(\mathbb{R}^3); \\ \partial_t u|_{t=0} = u_1 \in \dot{H}^{s-1}(\mathbb{R}^3). \end{cases} \quad (5)$$

Then we have

$$\begin{aligned} & \|(u(T), \partial_t u(T))\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|D_x^{\rho_1} u\|_{L^{q_1} L^{r_1}([0, T] \times \mathbb{R}^3)} \\ & \leq C \left(\|(u_0, u_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|D_x^{-\rho_2} F(x, t)\|_{L^{\bar{q}_2} L^{\bar{r}_2}([0, T] \times \mathbb{R}^3)} \right). \end{aligned}$$

The constant C does not depend on T .

Definition of Norms Fix $3 < p < 5$. We define the following norms with $s_p \leq s < 1$

$$\begin{aligned} \|v(x, t)\|_{S(I)} &= \|v(x, t)\|_{L^{2(p-1)} L^{2(p-1)}(I \times \mathbb{R}^3)}; \\ \|v(x, t)\|_{W(I)} &= \|v(x, t)\|_{L^4 L^4(I \times \mathbb{R}^3)}; \\ \|v(x, t)\|_{Z_s(I)} &= \|v(x, t)\|_{L^{\frac{2}{s+1}} L^{\frac{2}{2-s}}(I \times \mathbb{R}^3)}; \\ \|v(x, t)\|_{Y_s(I)} &= \|v(x, t)\|_{L^{\frac{2p}{s+1-(2p-2)(s-s_p)}} L^{\frac{2p}{2-s}}(I \times \mathbb{R}^3)}. \end{aligned}$$

Remark By the Strichartz estimates, we have if $u(x, t)$ is the solution of

$$\begin{cases} \partial_t^2 u - \Delta u = F(x, t), & (x, t) \in \mathbb{R}^3 \times \mathbb{R}; \\ u|_{t=0} = u_0 \in \dot{H}^s(\mathbb{R}^3); \\ \partial_t u|_{t=0} = u_1 \in \dot{H}^{s-1}(\mathbb{R}^3). \end{cases}$$

then

$$\begin{aligned} & \|(u(T), \partial_t u(T))\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|u\|_{Y_s([0, T])} \\ & \leq C \left(\|(u_0, u_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|F(x, t)\|_{Z_s([0, T])} \right). \end{aligned}$$

Definition of Solutions We say $u(t)(t \in I)$ is a solution of (1), if $(u, \partial_t u) \in C(I; \dot{H}^{s_p} \times \dot{H}^{s_p-1})$, with finite norms $\|u\|_{S(J)}$ and $\|D_x^{s_p-1/2} u\|_{W(J)}$ for any bounded closed interval $J \subseteq I$ so that the integral equation

$$u(t) = S(t)(u_0, u_1) + \int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau)) d\tau$$

holds for all time $t \in I$. Here $S(t)(u_0, u_1)$ is the solution of the linear wave equation with initial data (u_0, u_1) and

$$F(u) = \pm |u|^{p-1} u.$$

Remark We can take another way to define the solutions by substituting $S(I)$ and $W(I)$ norms by a single $Y_{s_p}(I)$ norm. Using the Strichartz estimates, these two definitions are equivalent to each other.

Local Theory By the Strichartz estimate and a fixed-point argument, we have the following theorems. (Please see [16] for more details)

Theorem 2.2. (Local solution) *For any initial data $(u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}$, there is a maximal interval $(-T_-(u_0, u_1), T_+(u_0, u_1))$ in which the equation has a solution.*

Theorem 2.3. (Scattering with small data) *There exists $\delta = \delta(p) > 0$ such that if the norm of the initial data $\|(u_0, u_1)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} < \delta$, then the Cauchy problem (1) has a global-in-time solution u with $\|u\|_{S(-\infty, +\infty)} < \infty$.*

Lemma 2.4. (Standard finite blow-up criterion) *If $T_+ < \infty$, then*

$$\|u\|_{S([0, T_+))} = \infty.$$

Theorem 2.5. (Long time perturbation theory) *(See [2, 9, 10, 11]) Let M, A, A' be positive constants. There exists $\varepsilon_0 = \varepsilon_0(M, A, A') > 0$ and $\beta > 0$ such that if $\varepsilon < \varepsilon_0$, then for any approximation solution \tilde{u} defined on $\mathbb{R}^3 \times I$ ($0 \in I$) and any initial data $(u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}$ satisfying*

$$(\partial_t^2 - \Delta)(\tilde{u}) - F(\tilde{u}) = e(x, t), \quad (x, t) \in \mathbb{R}^3 \times I;$$

$$\begin{cases} \sup_{t \in I} \|(\tilde{u}(t), \partial_t \tilde{u}(t))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} \leq A, \\ \|\tilde{u}\|_{S(I)} \leq M, \\ \|D_x^{s_p-1/2} \tilde{u}\|_{W(J)} < \infty \text{ for each } J \subset \subset I; \\ \|(u_0 - \tilde{u}(0), u_1 - \partial_t \tilde{u}(0))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} \leq A'; \end{cases} \quad (6)$$

$$\|D_x^{s_p-1/2} e\|_{L_I^{4/3} L_x^{4/3}} + \|S(t)(u_0 - \tilde{u}(0), u_1 - \partial_t \tilde{u}(0))\|_{S(I)} \leq \varepsilon;$$

there exists a solution of (1) defined in the interval I with the initial data (u_0, u_1) and satisfying

$$\|u\|_{S(I)} \leq C(M, A, A');$$

$$\sup_{t \in I} \|((u(t), \partial_t u(t)) - ((\tilde{u}(t), \partial_t \tilde{u}(t)))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} \leq C(M, A, A')(A' + \varepsilon + \varepsilon^\beta).$$

Theorem 2.6. (Perturbation theory with Y_{s_p} norm) Let M be a positive constant. There exists a constant $\varepsilon_0 = \varepsilon_0(M) > 0$, such that if $\varepsilon < \varepsilon_0$, then for any approximation solution \tilde{u} defined on $\mathbb{R}^3 \times I$ ($0 \in I$) and any initial data $(u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}$ satisfying

$$(\partial_t^2 - \Delta)(\tilde{u}) - F(\tilde{u}) = e(x, t), \quad (x, t) \in \mathbb{R}^3 \times I;$$

$$\|\tilde{u}\|_{Y_{s_p}(I)} < M; \quad \|(\tilde{u}(0), \partial_t \tilde{u}(0))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} < \infty;$$

$$\|e(x, t)\|_{Z_{s_p}(I)} + \|S(t)(u_0 - \tilde{u}(0), u_1 - \partial_t \tilde{u}(0))\|_{Y_{s_p}(I)} \leq \varepsilon;$$

there exists a solution $u(x, t)$ of (1) defined in the interval I with the initial data (u_0, u_1) and satisfying

$$\|u(x, t) - \tilde{u}(x, t)\|_{Y_{s_p}(I)} < C(M)\varepsilon.$$

$$\sup_{t \in I} \left\| \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} - \begin{pmatrix} \tilde{u}(t) \\ \partial_t \tilde{u}(t) \end{pmatrix} - S(t) \begin{pmatrix} u_0 - \tilde{u}(0) \\ u_1 - \partial_t \tilde{u}(0) \end{pmatrix} \right\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} < C(M)\varepsilon.$$

Remark If K is a compact subset of the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$, then there exists $T = T(K) > 0$ such that for any $(u_0, u_1) \in K$, $T_+(u_0, u_1) > T(K)$. This is a direct result from the perturbation theory.

2.2 Local Theory with more regular initial data

Let $s \in (s_p, 1]$. By a similar fixed argument we can obtain the following results.

Theorem 2.7. (Local solution with $\dot{H}^s \times \dot{H}^{s-1}$ initial data) If $(u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-1}$, then there is a maximal interval $(-T_-(u_0, u_1), T_+(u_0, u_1))$ in which the equation has a solution $u(x, t)$. In addition, we have

$$T_-(u_0, u_1), T_+(u_0, u_1) > T_1 \doteq C_{s,p}(\|(u_0, u_1)\|_{\dot{H}^s \times \dot{H}^{s-1}})^{-1/(s-s_p)};$$

$$\|u(x, t)\|_{Y_s([-T_1, T_1])} \leq C_{s,p}\|(u_0, u_1)\|_{\dot{H}^s \times \dot{H}^{s-1}}.$$

Theorem 2.8. (Weak long-time perturbation theory) *Let \tilde{u} be a solution of the equation (1) in the time interval $[0, T]$ with initial data $(\tilde{u}_0, \tilde{u}_1)$, so that*

$$\|(\tilde{u}_0, \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} < \infty; \quad \|\tilde{u}\|_{Y_s([0, T])} < M.$$

There exist two constants $\varepsilon_0(T, M), C(T, M) > 0$, such that if (u_0, u_1) is another pair of initial data with

$$\|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} < \varepsilon_0(T, M),$$

then there exists a solution u of the equation (1) in the time interval $[0, T]$ with initial data (u_0, u_1) so that

$$\|u - \tilde{u}\|_{Y_s([0, T])} \leq C(T, M) \|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}};$$

$$\sup_{t \in [0, T]} \|(u(t) - \tilde{u}(t), \partial_t u(t) - \partial_t \tilde{u}(t))\|_{\dot{H}^s \times \dot{H}^{s-1}} \leq C(T, M) \|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}}.$$

2.3 Notations and Technical Results

The Symbol \lesssim Throughout this paper, the inequality $A \lesssim B$ means that there exists a constant c , such that $A \leq cB$. In particular, a subscript of the symbol \lesssim implies that the constant c depends on the parameter(s) mentioned in the subscript but nothing else.

The Smooth Frequency Cutoff In this paper we use the notations $P_{<A}$ and $P_{>A}$ for the standard smooth frequency cutoff operators. In particular, we use the following notation on u for convenience.

$$u_{<A} \doteq P_{<A}u; \quad u_{>A} \doteq P_{>A}u.$$

Notation for Radial Functions If $u(x, t)$ is radial in the space, then $u(r, t)$ represents the value $u(x, t)$ when $|x| = r$.

Linear Wave Evolution Let $(u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^3)$ be a pair of initial data. Suppose $u(x, t)$ is the solution of the following linear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & (x, t) \in \mathbb{R}^3 \times \mathbb{R}; \\ u|_{t=0} = u_0; \\ \partial_t u|_{t=0} = u_1. \end{cases}$$

We will use the following notations to represent this solution u .

$$S(t_0)(u_0, u_1) = u(t_0);$$

$$S(t_0) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u(t_0) \\ \partial_t u(t_0) \end{pmatrix}.$$

Method of Center Cutoff Let $(v_0, v_1) \in \dot{H}^1 \times L^2(\mathbb{R}^3 \setminus B(0, r))$ be a pair of radial functions. We define ($R > r$)

$$(\Psi_R v_0)(x) = \begin{cases} v_0(x), & \text{if } |x| > R; \\ v_0(R), & \text{if } |x| \leq R. \end{cases}$$

$$(\Psi_R v_1)(x) = \begin{cases} v_1(x), & \text{if } |x| > R; \\ 0, & \text{if } |x| \leq R. \end{cases}$$

Lemma 2.9. Glue of \dot{H}^s Functions Let $-1 \leq s \leq 1$. Suppose $f(x)$ is a tempered distribution defined on \mathbb{R}^3 such that ($R > 0$)

$$f(x) = \begin{cases} f_1(x), & x \in B(0, 2R) \\ f_2(x), & x \in \mathbb{R}^3 \setminus B(0, R) \end{cases}$$

with $f_1, f_2 \in \dot{H}^s(\mathbb{R}^3)$. Then f is in the space $\dot{H}^s(\mathbb{R}^3)$ and

$$\|f\|_{\dot{H}^s(\mathbb{R}^3)} \leq C(s) \left(\|f_1\|_{\dot{H}^s(\mathbb{R}^3)} + \|f_2\|_{\dot{H}^s(\mathbb{R}^3)} \right).$$

Proof By a dilation we can always assume $R = 1$. Let $\phi(x)$ be a smooth, radial, nonnegative function such that

$$\phi(x) = \begin{cases} 1, & x \in B(0, 1) \\ 0, & x \in \mathbb{R}^3 \setminus B(0, 2) \end{cases}$$

Let us define a linear operator: $P(f) = \phi(x)f$. We know this operator is bounded from $\dot{H}^1(\mathbb{R}^3)$ to $\dot{H}^1(\mathbb{R}^3)$, and from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. Thus by an interpolation, this is a bounded operator from \dot{H}^s to itself if $0 < s < 1$. By duality P is also bounded from \dot{H}^s to itself if $-1 \leq s \leq 0$. In summary, P is a bounded operator from \dot{H}^s to itself for each $-1 \leq s \leq 1$. Now we have

$$f = Pf_1 + f_2 - Pf_2$$

as a tempered distribution. Thus

$$\|f\|_{\dot{H}^s} \leq \|Pf_1\|_{\dot{H}^s} + \|f_2\|_{\dot{H}^s} + \|Pf_2\|_{\dot{H}^s} \leq (\|P\|_s + 1)(\|f_1\|_{\dot{H}^s} + \|f_2\|_{\dot{H}^s}).$$

Lemma 2.10. Let $u(x, t)$ be a solution of the non-linear wave equation (1) with the condition (4), then

$$\left\| \begin{pmatrix} \int_{t_1}^{t_2} \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau)) d\tau \\ - \int_{t_1}^{t_2} \cos((\tau - t)\sqrt{-\Delta}) F(u(\tau)) d\tau \end{pmatrix} \right\|_{\dot{H}^{sp} \times \dot{H}^{sp-1}} \lesssim 1. \quad (7)$$

Proof Directly from the following identity.

$$\begin{pmatrix} \int_{t_1}^{t_2} \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau)) d\tau \\ - \int_{t_1}^{t_2} \cos((\tau - t)\sqrt{-\Delta}) F(u(\tau)) d\tau \end{pmatrix} = S(t - t_1) \begin{pmatrix} u(t_1) \\ \partial_t u(t_1) \end{pmatrix} - S(t - t_2) \begin{pmatrix} u(t_2) \\ \partial_t u(t_2) \end{pmatrix}. \quad (8)$$

Lemma 2.11. (Please see lemma 3.2 of [11]) Let $1/2 < s < 3/2$. If $u(y)$ is a radial $\dot{H}^s(\mathbb{R}^3)$ function, then

$$|u(y)| \lesssim_s \frac{1}{|y|^{\frac{3}{2}-s}} \|u\|_{\dot{H}^s}. \quad (9)$$

Remark This actually means that a radial \dot{H}^s function is uniformly continuous in $\mathbb{R}^3 \setminus B(0, R)$ if $R > 0$.

Lemma 2.12. Let $r_1, r_2 > 0$ and $t_0, t_1 \in \mathbb{R}$ so that $r_1 + r_2 \leq t_1 - t_0$. Suppose (u_0, u_1) is a weak limit in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$ as below

$$\begin{aligned} u_0 &= \lim_{T \rightarrow +\infty} \int_{t_1}^T \frac{\sin((t-t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(t) dt; \\ u_1 &= - \lim_{T \rightarrow +\infty} \int_{t_1}^T \cos((t-t_0)\sqrt{-\Delta}) F(t) dt. \end{aligned}$$

Here $F(x, t)$ is a function defined in $[t_1, \infty) \times \mathbb{R}^3$ with a finite $Z_{s_p}([t_1, T])$ norm for each $T > t_1$. In addition, we have $(1/2 < s_1 \leq 1, \chi$ is a characteristic function of the region indicated)

$$S = \|\chi_{|x| > r_2 + |t-t_1|}(x, t) F(x, t)\|_{L^1 L^{\frac{6}{5-2s_1}}([t_1, \infty) \times \mathbb{R}^3)} < +\infty. \quad (10)$$

Then there exists a pair $(\tilde{u}_0, \tilde{u}_1)$ with $\|(\tilde{u}_0, \tilde{u}_1)\|_{\dot{H}^{s_1} \times \dot{H}^{s_1-1}(\mathbb{R}^3)} \leq C_{s_1} S$ and

$$(u_0, u_1) = (\tilde{u}_0, \tilde{u}_1) \text{ in the ball } B(0, r_1).$$

Proof Let us define

$$\begin{aligned} u_{0,T} &= \int_{t_1}^T \frac{\sin((t-t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(t) dt; \\ u_{1,T} &= - \int_{t_1}^T \cos((t-t_0)\sqrt{-\Delta}) F(t) dt. \\ \tilde{u}_{0,T} &= \int_{t_1}^T \frac{\sin((t-t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}} (\chi F(t)) dt; \\ \tilde{u}_{1,T} &= - \int_{t_1}^T \cos((t-t_0)\sqrt{-\Delta}) (\chi F(t)) dt. \end{aligned}$$

By the Strichartz estimates and the assumption (10), we know the pair $(\tilde{u}_{0,T}, \tilde{u}_{1,T})$ converges strongly in $\dot{H}^{s_1} \times \dot{H}^{s_1-1}$ to a pair $(\tilde{u}_0, \tilde{u}_1)$ as $T \rightarrow +\infty$ so that

$$\|(\tilde{u}_0, \tilde{u}_1)\|_{\dot{H}^{s_1} \times \dot{H}^{s_1-1}(\mathbb{R}^3)} \leq C_{s_1} S.$$

In addition, we know the pair $(\tilde{u}_{0,T}, \tilde{u}_{1,T})$ is the same as $(u_{0,T}, u_{1,T})$ in the ball $B(0, r_1)$ by strong Huygens' principal. The figure 1 shows the region where the value of $F(x, t)$ may affect the value of the integrals in the ball $B(0, r_1)$. This region is disjoint with the cutoff area if $r_1 + r_2 \leq t_1 - t_0$. As a result, the pair $(\tilde{u}_{0,T}, \tilde{u}_{1,T})$ converges to (u_0, u_1) weakly in the ball $B(0, r_1)$ as the pair $(u_{0,T}, u_{1,T})$ does. Considering both the strong and weak convergence, we conclude

$$(u_0, u_1) = (\tilde{u}_0, \tilde{u}_1) \text{ in the ball } B(0, r_1).$$

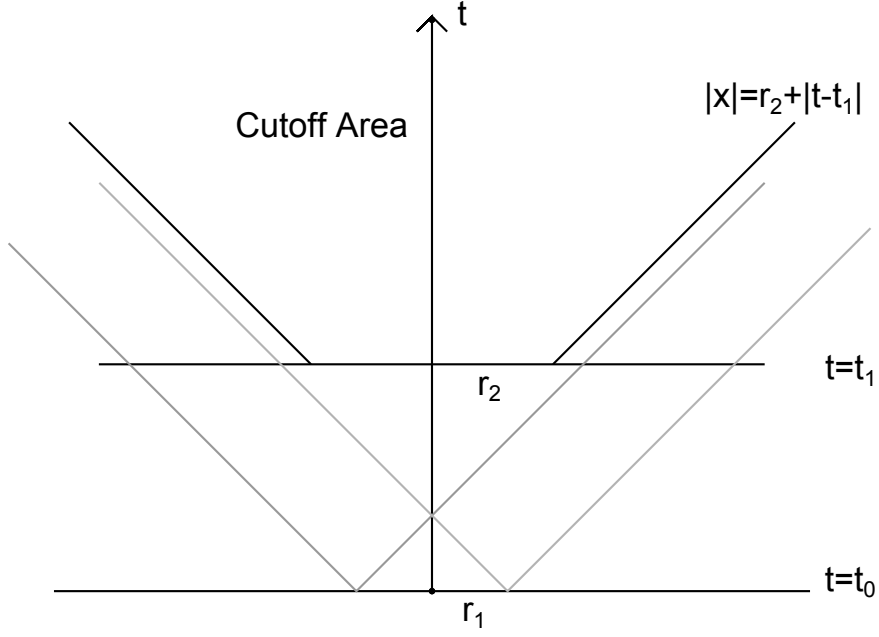


Figure 1: Illustration of Proof

3 Compactness Process

As we stated in the first section, the standard technique here is to show if the main theorem failed, there would be a special minimal blow-up solution. In addition, this solution is almost periodic modulo symmetries.

Definition A solution $u(x, t)$ of (1) is almost periodic modulo symmetries if there exists a positive function $\lambda(t)$ defined on its maximal lifespan I such that the set

$$\left\{ \left(\frac{1}{\lambda(t)^{3/2-s_p}} u \left(\frac{x}{\lambda(t)}, t \right), \frac{1}{\lambda(t)^{5/2-s_p}} \partial_t u \left(\frac{x}{\lambda(t)}, t \right) \right) : t \in I \right\}$$

is precompact in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$. The function $\lambda(t)$ is called the frequency scale function, because the solution $u(t)$ at time t concentrates around the frequency $\lambda(t)$ by the compactness.

Please note that here we use the radial condition, thus the only available symmetries are scalings. If we did not assume the radial condition, similar results would still hold but the symmetries would include translations besides scalings.

3.1 Existence of Minimal Blow-up Solution

Theorem 3.1. (Minimal blow-up solution) Assume that the main theorem failed. Then there would exist a solution $u(x, t)$ with a maximal lifespan I such that

$$\sup_{t \in I} \|(u(t), \partial_t u(t))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} < \infty,$$

u blows up in the positive direction at time $T_+ \leq +\infty$ with

$$\|u\|_{S([0, T_+))} = \infty.$$

In addition, u is almost periodic modulo scalings with a frequency scale function $\lambda(t)$. It is minimal in the following sense, if v is another solution with a maximal lifespan J and

$$\sup_{t \in J} \|(v(t), \partial_t v(t))\|_{\dot{H}^{sp} \times \dot{H}^{sp-1}} < \sup_{t \in I} \|(u(t), \partial_t u(t))\|_{\dot{H}^{sp} \times \dot{H}^{sp-1}},$$

then v is a global solution in time and scatters.

The main tool to obtain this result is the profile decomposition. One could follow the argument in [12] in order to find a proof. In that paper C.E.Kenig and F.Merle deal with the cubic defocusing NLS under similar assumptions.

3.2 Three enemies

Since the frequency scale function $\lambda(t)$ plays an important role in the further discussion, it is helpful if we could make additional assumptions on this function. It turns out that we could reduce the whole problem into the following three special cases. This method of three enemies was introduced in R.Killip, T.Tao and M.Visan's paper [15].

Theorem 3.2. (Three enemies) *Suppose our main theorem failed, then there would exist a minimal blow-up solution u satisfying all the conditions we mentioned in the previous theorem, so that one of the following three assumptions on its lifespan I and frequency scale function $\lambda(t)$ holds*

- (I) **(Soliton-like case)** $I = \mathbb{R}$ and $\lambda(t) = 1$.
- (II) **(High-to-low frequency cascade)** $I = \mathbb{R}$, $\lambda(t) \leq 1$ and

$$\liminf_{t \rightarrow \pm\infty} \lambda(t) = 0.$$

- (III) **(Self-similar case)** $I = \mathbb{R}^+$ and $\lambda(t) = 1/t$.

Please note that the minimal blow-up solution u here could be different from the one we found in the previous theorem. But we can always manufacture a minimal blow-up solution in one of these three cases from the original one. One can follow the method used in the paper [15] to verify this theorem.

3.3 Further Compactness Results

Fix a radial cutoff function $\varphi(x) \in C^\infty(\mathbb{R}^3)$ with the following properties.

$$\varphi(x) \begin{cases} = 0, & |x| \leq 1/2; \\ \in [0, 1], & 1/2 \leq |x| \leq 1; \\ = 1, & |x| \geq 1. \end{cases}$$

Given a minimal blow-up solution u mentioned above and its frequency scale function $\lambda(t)$, we have the following propositions by a compactness argument.

Proposition 3.3. *Let u be a minimal blow-up solution with a maximal lifespan I as above. There exist constants $d, C' > 0$ and $C_1 > 1$ independent of t such that*

(i) *The interval $[t - d\lambda^{-1}(t), t + d\lambda^{-1}(t)] \subseteq I$ for all $t \in I$. In addition, for each $t' \in [t - d\lambda^{-1}(t), t + d\lambda^{-1}(t)]$, we have*

$$\frac{1}{C_1}\lambda(t) \leq \lambda(t') \leq C_1\lambda(t). \quad (11)$$

(ii) *The following estimate holds for each s_p -admissible pair (q, r) and each $t \in I$.*

$$\|u\|_{L^q L^r([t - d\lambda^{-1}(t), t + d\lambda^{-1}(t)] \times \mathbb{R}^3)} \leq C'.$$

Proposition 3.4. *Given $\varepsilon > 0$, there exists $R_1 = R_1(\varepsilon) > 0$, such that the following inequality holds for each $t \in I$.*

$$\left\| \left(\varphi \left(\frac{x}{R_1 \lambda^{-1}(t)} \right) u(t), \varphi \left(\frac{x}{R_1 \lambda^{-1}(t)} \right) \partial_t u(t) \right) \right\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)} \leq \varepsilon.$$

Proposition 3.5. *There exists two constants $R_0, \eta_0 > 0$, such that the following inequality holds for each $t \in I$. (The constant d is the same constant we used in proposition 3.3)*

$$\int_t^{t+d\lambda^{-1}(t)} \int_{|x| < R_0 \lambda^{-1}(t)} \frac{|u(x, \tau)|^{p+1}}{|x|} dx d\tau \geq \lambda(t)^{2-2s_p} \eta_0.$$

Proof By a compactness argument we obtain that there exist $R_0, \eta_0 > 0$, so that for all $t \in I$,

$$\int_0^d \int_{|x| < R_0} \frac{(\frac{1}{\lambda(t)^{2/(p-1)}} |u(\lambda^{-1}(t)x, \lambda^{-1}(t)\tau + t)|)^{p+1}}{|x|} dx d\tau \geq \eta_0.$$

This implies

$$\begin{aligned} \int_0^d \int_{|x| < R_0} \frac{|u(\lambda^{-1}(t)x, \lambda^{-1}(t)\tau + t)|^{p+1}}{\lambda^{-1}(t)|x|} \frac{dx d\tau}{\lambda(t)^{\frac{2(p+1)}{p-1}+1}} &\geq \eta_0. \\ \frac{1}{\lambda(t)^{4/(p-1)-1}} \int_0^d \int_{|x| < R_0} \frac{|u(\lambda^{-1}(t)x, \lambda^{-1}(t)\tau + t)|^{p+1}}{\lambda^{-1}(t)|x|} \frac{dx d\tau}{\lambda(t)^4} &\geq \eta_0. \\ \int_t^{t+d\lambda^{-1}(t)} \int_{|x| < R_0 \lambda^{-1}(t)} \frac{|u(x, \tau)|^{p+1}}{|x|} dx d\tau &\geq \lambda(t)^{4/(p-1)-1} \eta_0 \\ &= \lambda(t)^{2-2s_p} \eta_0. \end{aligned} \quad (12)$$

3.4 The Duhamel Formula

The following Duhamel formula will be frequently used in later sections.

Proposition 3.6. (The Duhamel formula) *Let u be a minimal blow-up solution described above with a maximal lifespan $I = (T_-, \infty)$. Then we have*

$$\begin{aligned} u(t) &= \lim_{T \rightarrow +\infty} \int_t^T \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau)) d\tau; \\ \partial_t u(t) &= - \lim_{T \rightarrow +\infty} \int_t^T \cos((\tau - t)\sqrt{-\Delta}) F(u(\tau)) d\tau. \end{aligned}$$

$$\begin{aligned}
u(t) &= \lim_{T \rightarrow T_-} \int_T^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau)) d\tau; \\
\partial_t u(t) &= \lim_{T \rightarrow T_-} \int_T^t \cos((t-\tau)\sqrt{-\Delta}) F(u(\tau)) d\tau.
\end{aligned}$$

Given a time $t \in I$, these limits are weak limits in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$. If J is a closed interval compactly supported in I , then one could also understand the formula for $u(t)$ as a strong limit in the space $L^q L^r(J \times \mathbb{R}^3)$, as long as (q, r) is an s_p -admissible pair with $q \neq \infty$.

Remark Actually we have

$$\begin{pmatrix} \int_t^T \frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau)) d\tau \\ - \int_t^T \cos((\tau-t)\sqrt{-\Delta}) F(u(\tau)) d\tau \end{pmatrix} = \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} - S(t-T) \begin{pmatrix} u(T) \\ \partial_t u(T) \end{pmatrix}. \quad (13)$$

Thus we only need to show the corresponding limit of the last term is zero in order to verify this formula. Please see lemma 10.2 in the appendix for details.

4 Energy Estimate near Infinity

In this section, we will prove the following theorem for a minimal blow-up solution $u(x, t)$. The method was previously used in the supercritical case of the equation. (please see [11] for more details) In the supercritical case, by the Sobolev embedding, the energy automatically exists at least locally in the space, for any given time $t \in I$. In the subcritical case, however, we need to use the approximation techniques.

Theorem 4.1. (Energy estimate near infinity) *Let $u(x, t)$ be a minimal blow-up solution as we found in the previous section. Then $(u(x, t), \partial_t u(x, t)) \in \dot{H}^1 \times L^2(\mathbb{R} \setminus B(0, r))$ for each $r > 0$, $t \in I$. Actually we have*

$$\int_{r < |x| < 4r} (|\nabla u(x, t)|^2 + |\partial_t u(x, t)|^2) dx \lesssim r^{-2(1-s_p)}. \quad (14)$$

4.1 Preliminary Results

Introduction to $w(r, t)$ Let $u(x, t)$ be a radial solution of the wave equation

$$\partial_t^2 u - \Delta u = F(x, t).$$

If we define $w(r, t), h(r, t) : \mathbb{R}^+ \times I \rightarrow \mathbb{R}$ so that

$$w(|x|, t) = |x|u(x, t); \quad h(|x|, t) = |x|F(x, t);$$

then we have $w(r, t)$ is the solution of the one-dimensional wave equation

$$\partial_t^2 w - \partial_r^2 w = h(r, t).$$

Lemma 4.2. *Let $(u(x, t_0), \partial_t u(x, t_0))$ be radial and in the energy space $\dot{H}^1 \times L^2$ locally, then for any $0 < a < b < \infty$, we have the identity*

$$\frac{1}{4\pi} \int_{a < |x| < b} (|\nabla u|^2 + |\partial_t u|^2) dx = \left(\int_a^b [(\partial_r w)^2 + (\partial_t w)^2] dr \right) + (au^2(a) - bu^2(b))$$

holds (if we take the value of the functions at time t_0).

Proof By direct computation

$$\begin{aligned} \int_a^b [(\partial_r w)^2 + (\partial_t w)^2] dr &= \int_a^b [(r\partial_r u + u)^2 + (r\partial_t u)^2] dr \\ &= \int_a^b [r^2(\partial_r u)^2 + u^2 + r^2(\partial_t u)^2] dr + \int_a^b 2ru\partial_r u dr \\ &= \int_a^b [r^2(\partial_r u)^2 + r^2(\partial_t u)^2 + u^2] dr + \int_a^b r d(u^2) \\ &= \int_a^b r^2[(\partial_r u)^2 + (\partial_t u)^2] dr + [ru^2]_a^b \\ &= \frac{1}{4\pi} \int_{a < |x| < b} (|\nabla u|^2 + |\partial_t u|^2) dx + bu^2(b) - au^2(a). \end{aligned}$$

Lemma 4.3. *Let $w(r, t)$ be a solution to the following equation for $(r, t) \in \mathbb{R}^+ \times I$*

$$\partial_t^2 w - \partial_r^2 w = h(r, t),$$

so that $(w, \partial_t w) \in C(I; \dot{H}^1 \times L^2(R_1 < r < R_2))$ for any $0 < R_1 < R_2 < \infty$. Let us define

$$z_1(r, t) = \partial_t w(r, t) - \partial_r w(r, t);$$

$$z_2(r, t) = \partial_t w(r, t) + \partial_r w(r, t).$$

Then we have ($M > 0$)

$$\begin{aligned} &\left| \left(\int_{r_0}^{4r_0} |z_1(r, t_0)|^2 dr \right)^{1/2} - \left(\int_{r_0+M}^{4r_0+M} |z_1(r, t_0 + M)|^2 dr \right)^{1/2} \right| \\ &\leq \left(\int_{r_0}^{4r_0} \left(\int_0^M h(r+t, t_0+t) dt \right)^2 dr \right)^{1/2}, \end{aligned} \quad (15)$$

if $t_0, t_0 + M \in I$;

$$\begin{aligned} &\left| \left(\int_{r_0}^{4r_0} |z_2(r, t_0)|^2 dr \right)^{1/2} - \left(\int_{r_0+M}^{4r_0+M} |z_2(r, t_0 - M)|^2 dr \right)^{1/2} \right| \\ &\leq \left(\int_{r_0}^{4r_0} \left(\int_0^M h(r+t, t_0-t) dt \right)^2 dr \right)^{1/2}, \end{aligned} \quad (16)$$

if $t_0, t_0 - M \in I$.

Proof We will assume w has sufficient regularity, otherwise we only need to use the standard techniques of smooth approximation. Let us define

$$z(r, s) = (\partial_t - \partial_r)w(r + s, t_0 + s).$$

We have

$$\partial_s z(r, s) = (\partial_t + \partial_r)(\partial_t - \partial_r)w(r + s, t_0 + s) = h(r + s, t_0 + s).$$

Thus

$$z(r, M) = z(r, 0) + \int_0^M h(r + t, t_0 + t) dt.$$

Applying the triangle inequality, we obtain the first inequality. The second inequality could be proved in a similar way.

4.2 Smooth approximation

Let $u(x, t)$ be a minimal blow-up solution. Choose a smooth, nonnegative, radial function $\varphi(x, t)$ supported in the four-dimensional ball $B(0, 1) \subset \mathbb{R}^4$ such that

$$\int_{\mathbb{R}^3 \times \mathbb{R}} \varphi(x, t) dx dt = 1.$$

Let d be the number given in proposition 3.3. If $\varepsilon < d$, we define (both the functions u and $F(u)$ are locally integrable)

$$\varphi_\varepsilon(x, t) = \frac{1}{\varepsilon^4} \varphi(x/\varepsilon, t/\varepsilon);$$

$$u_\varepsilon = u * \varphi_\varepsilon; \quad F_\varepsilon = F(u) * \varphi_\varepsilon.$$

This makes $u_\varepsilon(x, t)$ be a smooth solution of the linear wave equation

$$\partial_t^2 u_\varepsilon(x, t) - \Delta u_\varepsilon(x, t) = F_\varepsilon(x, t).$$

with the convergence (using the continuity of $(u(t), \partial_t u(t))$ in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$)

$$(u_\varepsilon(t_0), \partial_t u_\varepsilon(t_0)) \rightarrow (u(t_0), \partial_t u(t_0)) \text{ in the space } \dot{H}^{s_p} \times \dot{H}^{s_p-1} \text{ for each } t_0 \in I$$

and

$$\|(u_\varepsilon(t_0), \partial_t u_\varepsilon(t_0))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} \lesssim 1.$$

In addition, if $a - \varepsilon \in I$, we have

$$\|F_\varepsilon(x, t)\|_{Z_{s_p}([a, b])} < \infty.$$

Lemma 4.4. (Almost periodic property) *The following set*

$$\left\{ \left(\frac{1}{\lambda(t)^{3/2-s_p}} u_\varepsilon \left(\frac{x}{\lambda(t)}, t \right), \frac{1}{\lambda(t)^{5/2-s_p}} \partial_t u_\varepsilon \left(\frac{x}{\lambda(t)}, t \right) \right) : t \in [d+1, \infty) \right\}$$

is precompact in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$ for each fixed $\varepsilon < d$.

Stretch of the proof Given a sequence $\{t_n\}$, WLOG, we could assume

$$\lambda(t_n) \rightarrow \lambda_0 \in [0, 1];$$

$$\left(\frac{1}{\lambda(t_n)^{3/2-s_p}} u \left(\frac{x}{\lambda(t_n)}, t_n \right), \frac{1}{\lambda(t_n)^{5/2-s_p}} \partial_t u \left(\frac{x}{\lambda(t_n)}, t_n \right) \right) \rightarrow (u_0, u_1);$$

by extracting a subsequence if necessary. Let $\tilde{u}(x, t)$ be the solution of the equation (1) with initial data (u_0, u_1) . By the long-time perturbation theory we know

$$\sup_{t \in [-d, d]} \left\| \begin{pmatrix} \frac{1}{\lambda(t_n)^{3/2-s_p}} u \left(\frac{x}{\lambda(t_n)}, t_n + \frac{t}{\lambda(t_n)} \right) \\ \frac{1}{\lambda(t_n)^{5/2-s_p}} \partial_t u \left(\frac{x}{\lambda(t_n)}, t_n + \frac{t}{\lambda(t_n)} \right) \end{pmatrix} - \begin{pmatrix} \tilde{u}(t) \\ \partial_t \tilde{u}(t) \end{pmatrix} \right\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} \rightarrow 0.$$

This implies

$$\begin{aligned} \begin{pmatrix} \frac{1}{\lambda(t_n)^{3/2-s_p}} u_\varepsilon \left(\frac{x}{\lambda(t_n)}, t_n \right) \\ \frac{1}{\lambda(t_n)^{5/2-s_p}} \partial_t u_\varepsilon \left(\frac{x}{\lambda(t_n)}, t_n \right) \end{pmatrix} &= \begin{bmatrix} \varphi_{\varepsilon \lambda(t_n)} * \begin{pmatrix} \frac{1}{\lambda(t_n)^{3/2-s_p}} u \left(\frac{x}{\lambda(t_n)}, t_n + \frac{t}{\lambda(t_n)} \right) \\ \frac{1}{\lambda(t_n)^{5/2-s_p}} \partial_t u \left(\frac{x}{\lambda(t_n)}, t_n + \frac{t}{\lambda(t_n)} \right) \end{pmatrix} \end{bmatrix} (x, 0) \\ &= \begin{bmatrix} \varphi_{\varepsilon \lambda(t_n)} * \begin{pmatrix} \tilde{u} \\ \partial_t \tilde{u} \end{pmatrix} \end{bmatrix} (x, 0) + o(1) \\ &= \begin{cases} \begin{bmatrix} \varphi_{\varepsilon \lambda_0} * \begin{pmatrix} \tilde{u} \\ \partial_t \tilde{u} \end{pmatrix} \end{bmatrix} (x, 0) + o(1) & \text{if } \lambda_0 > 0; \\ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} + o(1) & \text{if } \lambda_0 = 0; \end{cases} \end{aligned}$$

Remark The error $o(1)$ tends to zero as $n \rightarrow \infty$ in the sense of the $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$ norm.

The Duhamel Formula By the almost periodic property above we know the following Duhamel formula still holds for u_ε in the sense of weak limit if $t_0 - \varepsilon \in I$.

$$\begin{aligned} u_\varepsilon(t_0) &= \int_{t_0}^{+\infty} \frac{\sin((\tau - t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}} F_\varepsilon(x, \tau) d\tau; \\ \partial_t u_\varepsilon(t_0) &= - \int_{t_0}^{+\infty} \cos((\tau - t_0)\sqrt{-\Delta}) F_\varepsilon(x, \tau) d\tau. \end{aligned}$$

In the soliton-like or high-to-low frequency cascade case, we can also verify the Duhamel formula in the negative time direction.

The idea to prove theorem 4.1 If we could obtain the following estimate

$$\int_{r < |x| < 4r} (|\nabla u_\varepsilon(x, t_0)|^2 + |\partial_t u_\varepsilon(x, t_0)|^2) dx \leq C r^{-2(1-s_p)}, \quad (17)$$

so that the constant C is independent of $r > 0$, $t_0 \in I$ and $\varepsilon < \varepsilon_0(r, t_0)$, then we would be able to prove theorem 4.1 by letting ε converge to zero. One could read lemma 10.5 if interested in the details of this argument.

Remark We have to apply the smooth kernel on the whole non-linear term. Because if we just made the initial data smooth, we would not resume the compactness conditions of the minimal blow-up solution.

Lemma 4.5. *If $|x| > 10\varepsilon$, we have*

$$|u_\varepsilon(x, t)| \leq \frac{C}{|x|^{2/(p-1)}}; \quad |F_\varepsilon(x, t)| \leq \frac{C}{|x|^{2p/(p-1)}}.$$

The constant C depends only on the upper bound $\sup_{t \in I} \|(u, \partial_t u)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}}$.

Proof This comes from the estimate (9) and an easy computation.

4.3 Uniform Estimate on u_ε

In this subsection, we will prove the following lemma. It implies theorem 4.1 immediately by our argument above. The functions $w_\varepsilon(r, t)$ and $z_{i,\varepsilon}(r, t)$ below are defined as described earlier using $u_\varepsilon(x, t)$.

Lemma 4.6. *Let $t_0 \in I$ and $r_0 > 0$, then for sufficiently small ε , we have*

$$\int_{r_0 < |x| < 4r_0} (|\nabla u_\varepsilon(x, t_0)|^2 + |\partial_t u_\varepsilon(x, t_0)|^2) dx \leq C r_0^{-2(1-s_p)}. \quad (18)$$

The constant C could be chosen independent of t_0, r_0 and ε .

Step 1 Conversion to $w_\varepsilon(r, t)$ First choose $\varepsilon < \min\{r_0/10, d\}$. If the minimal blow-up solution is a self-similar one, we also require $\varepsilon < t_0/2$. Let us apply lemma 4.2 and lemma 4.5. It is sufficient to show

$$\int_{r_0}^{4r_0} (|\partial_r w_\varepsilon(r, t_0)|^2 + |\partial_t w_\varepsilon(r, t_0)|^2) dr \leq C r_0^{-2(1-s_p)}.$$

In other words,

$$\int_{r_0}^{4r_0} (|z_{1,\varepsilon}(r, t_0)|^2 + |z_{2,\varepsilon}(r, t_0)|^2) dr \leq C r_0^{-2(1-s_p)}. \quad (19)$$

Step 2 Expansion of $z_{1,\varepsilon}$ Let us break $(u_\varepsilon(t), \partial_t u_\varepsilon(t))$ into two pieces.

$$\begin{aligned} u_\varepsilon^{(1)}(t) &= \int_t^{t_0+100r_0} \frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F_\varepsilon(x, \tau) d\tau; \\ \partial_t u_\varepsilon^{(1)}(t) &= - \int_t^{t_0+100r_0} \cos((\tau-t)\sqrt{-\Delta}) F_\varepsilon(x, \tau) d\tau. \end{aligned}$$

$$\begin{aligned} u_\varepsilon^{(2)}(t) &= \int_{t_0+100r_0}^\infty \frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F_\varepsilon(x, \tau) d\tau; \\ \partial_t u_\varepsilon^{(2)}(t) &= - \int_{t_0+100r_0}^\infty \cos((\tau-t)\sqrt{-\Delta}) F_\varepsilon(x, \tau) d\tau. \end{aligned}$$

These are smooth functions and we have

$$(u_\varepsilon(x, t_0), \partial_t u_\varepsilon(x, t_0)) = (u_\varepsilon^{(1)}(x, t_0), \partial_t u_\varepsilon^{(1)}(x, t_0)) + (u_\varepsilon^{(2)}(x, t_0), \partial_t u_\varepsilon^{(2)}(x, t_0)).$$

Defining $w_\varepsilon^{(j)}, z_{1,\varepsilon}^{(j)}$ accordingly for $j = 1, 2$, we have

$$z_{1,\varepsilon}(x, t_0) = z_{1,\varepsilon}^{(1)}(x, t_0) + z_{1,\varepsilon}^{(2)}(x, t_0).$$

Step 3 Short-time Contribution we have $u_\varepsilon^{(1)}$ satisfies the wave equation

$$\begin{cases} \partial_t^2 u_\varepsilon^{(1)} - \Delta u_\varepsilon^{(1)} = F_\varepsilon(x, t), & (x, t) \in \mathbb{R}^3 \times (t_0^-, +\infty); \\ u_\varepsilon^{(1)}|_{t=t_0+100r_0} = 0 \in \dot{H}^{s_p}(\mathbb{R}^3); \\ \partial_t u_\varepsilon^{(1)}|_{t=t_0+100r_0} = 0 \in \dot{H}^{s_p-1}(\mathbb{R}^3). \end{cases}$$

Thus $w_\varepsilon^{(1)}$ is a smooth solution of

$$\begin{cases} \partial_t^2 w_\varepsilon^{(1)} - \partial_r^2 w_\varepsilon^{(1)} = r F_\varepsilon(r, t), & (r, t) \in \mathbb{R}^+ \times (t_0^-, +\infty); \\ w_\varepsilon^{(1)}|_{t=t_0+100r_0} = 0; \\ \partial_t w_\varepsilon^{(1)}|_{t=t_0+100r_0} = 0. \end{cases}$$

Applying lemma 4.3 and lemma 4.5, we obtain

$$\begin{aligned} \left(\int_{r_0}^{4r_0} |z_{1,\varepsilon}^{(1)}(r, t_0)|^2 dr \right)^{1/2} &\leq \left(\int_{r_0}^{4r_0} \left(\int_0^{100r_0} (t+r) F_\varepsilon(t+r, t+t_0) dt \right)^2 dr \right)^{1/2} \\ &\lesssim \left(\int_{r_0}^{4r_0} \left(\int_0^{100r_0} (t+r) \frac{1}{(t+r)^{\frac{2p}{p-1}}} dt \right)^2 dr \right)^{1/2} \\ &\lesssim \left(\int_{r_0}^{4r_0} \left(\int_0^{100r_0} \frac{1}{(t+r)^{1+\frac{2}{p-1}}} dt \right)^2 dr \right)^{1/2} \\ &\lesssim \left(\int_{r_0}^{4r_0} \frac{1}{r^{4/(p-1)}} dr \right)^{1/2} \\ &\lesssim \frac{1}{r_0^{1-s_p}}. \end{aligned}$$

Step 4 Long-time Contribution Let us define a cutoff function $\chi(x, t)$ to be the characteristic function of the region $\{(x, t) : |x| > t - t_0 - 50r_0\}$. By lemma 4.5, we know

$$\begin{aligned} \|\chi F_\varepsilon\|_{L^1 L^2([t_0+100r_0, \infty) \times \mathbb{R}^3)} &= \int_{t_0+100r_0}^{\infty} \left(\int_{|x| > t-t_0-50r_0} |F_\varepsilon|^2 dx \right)^{1/2} dt \\ &\lesssim \int_{t_0+100r_0}^{\infty} \left(\int_{|x| > t-t_0-50r_0} \frac{1}{|x|^{4p/(p-1)}} dx \right)^{1/2} dt \end{aligned}$$

$$\begin{aligned}
&\lesssim \int_{t_0+100r_0}^{\infty} \left(\frac{1}{|t-t_0-50r_0|^{1+4/(p-1)}} \right)^{1/2} dt \\
&\lesssim \int_{t_0+100r_0}^{\infty} \frac{1}{|t-t_0-50r_0|^{\frac{1}{2}+\frac{2}{p-1}}} dt \\
&\lesssim \frac{1}{r_0^{1-s_p}}
\end{aligned}$$

Applying lemma 2.12, we obtain

$$\int_{r_0 < |x| < 4r_0} (|\nabla u_{\varepsilon}^{(2)}(x, t_0)|^2 + |\partial_t u_{\varepsilon}^{(2)}(x, t_0)|^2) dx \lesssim r_0^{2(s_p-1)}.$$

Applying lemma 4.2 and using the fact (plus (9))

$$\begin{aligned}
&\|(u_{\varepsilon}^{(2)}(t_0), \partial_t u_{\varepsilon}^{(2)}(t_0))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} \\
&= \left\| S(-100r_0) \begin{pmatrix} u_{\varepsilon}(t_0 + 100r_0) \\ \partial_t u_{\varepsilon}(t_0 + 100r_0) \end{pmatrix} \right\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} \\
&= \|(u_{\varepsilon}(t_0 + 100r_0), \partial_t u_{\varepsilon}(t_0 + 100r_0))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} \\
&\leq \sup_I \|(u, \partial_t u)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} \lesssim 1,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\int_{r_0}^{4r_0} (|\partial_r w_{\varepsilon}^{(2)}(r, t_0)|^2 + |\partial_t w_{\varepsilon}^{(2)}(r, t_0)|^2) dr \lesssim r_0^{2(s_p-1)}. \\
&\int_{r_0}^{4r_0} |z_{1,\varepsilon}^{(2)}(r, t_0)|^2 dr \lesssim r_0^{2(s_p-1)}.
\end{aligned}$$

Combining with the estimate for $z_{1,\varepsilon}^{(1)}$, we have

$$\int_{r_0}^{4r_0} |z_{1,\varepsilon}(r, t_0)|^2 dr \lesssim r_0^{2(s_p-1)}.$$

Step 5 Estimate of $z_{2,\varepsilon}$ We also need to consider $z_{2,\varepsilon}$. In the soliton-like case or the high-to-low frequency cascade case, this could be done in exactly the same way as $z_{1,\varepsilon}$. Now let us consider the self-similar case.

Lemma 4.7. *Let u be a self-similar minimal blow-up solution. If $t_0 \leq 0.3r_0$, Then $(u(t_0), \partial_t u(t_0))$ is in $\dot{H}^1 \times L^2(|x| > 0.9r_0)$ with*

$$\int_{|x| > 0.9r_0} (|\nabla u(x, t_0)|^2 + |\partial_t u(x, t_0)|^2) dx \lesssim r_0^{2(s_p-1)}.$$

Proof We have (the Duhamel Formula)

$$\begin{aligned}
u(t_0) &= \int_{0^+}^{t_0} \frac{\sin((t_0 - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(x, t) dt; \\
\partial_t u(t_0) &= \int_{0^+}^{t_0} \cos((t_0 - t)\sqrt{-\Delta}) F(x, t) dt.
\end{aligned}$$

$$\begin{aligned}\tilde{u}_0 &= \int_{0^+}^{t_0} \frac{\sin((t_0 - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} \chi(|x| > 0.5r_0) F(x, t) dt; \\ \tilde{u}_1 &= \int_{0^+}^{t_0} \cos((t_0 - t)\sqrt{-\Delta}) \chi(|x| > 0.5r_0) F(x, t) dt.\end{aligned}$$

A straightforward computation shows $\|\chi F\|_{L^1 L^2((0^+, t_0) \times \mathbb{R}^3)} \lesssim r_0^{s_p-1}$. This means $(\tilde{u}_0, \tilde{u}_1)$ is in the space $\dot{H}^1 \times L^2(\mathbb{R}^3)$ with a norm $\lesssim r_0^{s_p-1}$. By strong Huygens's principal we can repeat the argument we used in lemma 2.12 and obtain

$$(u(t_0), \partial_t u(t_0)) = (\tilde{u}_0, \tilde{u}_1) \text{ in the region } \mathbb{R}^3 \setminus B(0, 0.9r_0).$$

This completes the proof.

Lemma 4.8. *Let u be a self-similar solution. If $t_0 \leq 0.2r_0$ and $\varepsilon < t_0/2$, then we have*

$$\int_{r_0 < |x| < 4r_0} (|\nabla u_\varepsilon(x, t_0)|^2 + |\partial_t u_\varepsilon(x, t_0)|^2) dx \lesssim r_0^{2(s_p-1)}.$$

Proof We have $\nabla u_\varepsilon = \varphi_\varepsilon * \nabla u$, thus $|\nabla u_\varepsilon| \leq \varphi_\varepsilon * |\nabla u|$. Thus (we have $\varepsilon < 0.1r_0$)

$$\int_{r_0 < |x| < 4r_0} |\nabla u_\varepsilon(x, t_0)|^2 dx \leq \sup_{t \in [t_0 - \varepsilon, t_0 + \varepsilon]} \int_{0.9r_0 < |x| < 4.1r_0} |\nabla u(x, t)|^2 dx \lesssim r_0^{2(s_p-1)}$$

by our previous lemma. The other term could be estimated using the same way.

Remark By lemma 4.2 and lemma 4.5, this lemma implies (if $t_0 \leq 0.2r_0$)

$$\int_{r_0}^{4r_0} (|\partial_r w_\varepsilon(r, t_0)|^2 + |\partial_t w_\varepsilon(r, t_0)|^2) dr \lesssim r_0^{2(s_p-1)}. \quad (20)$$

In the self-similar case, let us recall that we always choose $\varepsilon < \min\{r_0/10, t_0/2, d\}$. By lemma 4.8 and its remark, we only need to consider the case $t_0 > 0.2r_0$ in order to estimate $z_{2,\varepsilon}$. Applying lemma 4.3, we have

$$\begin{aligned}\left(\int_{r_0}^{4r_0} |z_{2,\varepsilon}(r, t_0)|^2 dr \right)^{1/2} &\leq \left(\int_{t_0+0.8r_0}^{t_0+3.8r_0} |z_{2,\varepsilon}(r, 0.2r_0)|^2 dr \right)^{1/2} \\ &\quad + \left(\int_{r_0}^{4r_0} \left(\int_0^{t_0-0.2r_0} (t+r) F_\varepsilon(t+r, t_0-t) dt \right)^2 dr \right)^{1/2}\end{aligned}$$

The first term is dominated by $r_0^{s_p-1}$ because of (20). We can gain the same upper bound for the second term by a basic computation similar to the one we used for $z_{1,\varepsilon}$.

Step 6 Conclusion Now we combine the estimates for $z_{1,\varepsilon}$ and $z_{2,\varepsilon}$ thus conclude our lemma 4.6 and theorem 4.1. Applying lemma 4.2, we obtain

Proposition 4.9. *Let $u(x, t)$ be a minimal blow-up solution as above, we have*

$$\begin{aligned}\int_{r_0}^{4r_0} (|\partial_r w(r, t_0)|^2 + |\partial_t w(r, t_0)|^2) dr &\lesssim r_0^{2(s_p-1)}. \\ \int_{r_0}^{4r_0} (|z_1(r, t_0)|^2 + |z_2(r, t_0)|^2) dr &\lesssim r_0^{2(s_p-1)}.\end{aligned}$$

5 Recurrence Process

Starting Point Let $u(x, t)$ be a minimal blow-up solution of (1) as we obtained in the section of compactness process with a frequency scale function $\lambda(t)$. In addition, the following set is precompact in the space $\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^3)$ for some $s \in [s_p, 1)$.

$$\left\{ \left(\frac{1}{\lambda(t)^{3/2-s_p}} u \left(\frac{x}{\lambda(t)}, t \right), \frac{1}{\lambda(t)^{5/2-s_p}} \partial_t u \left(\frac{x}{\lambda(t)}, t \right) \right) : t \in I \right\}$$

In this section we will try to gain higher regularity than $\dot{H}^s \times \dot{H}^{s-1}$ by assuming the conditions above.

5.1 Setup and Technical Lemmas

Definition Let us define

$$\begin{aligned} S(A) &= \sup_{t \in I} (\lambda(t))^{s_p-s} \|u_{>\lambda(t)A}\|_{Y_s([t, t+d\lambda^{-1}(t)])}; \\ N(A) &= \sup_{t \in I} (\lambda(t))^{s_p-s} \|P_{>\lambda(t)A} F(u)\|_{Z_s([t, t+d\lambda^{-1}(t)])}. \end{aligned}$$

By our assumptions on compactness and proposition 3.3, we have

$$\left\{ \left(\frac{1}{\lambda(t)^{3/2-s_p}} u \left(\frac{x}{\lambda(t)}, t + \frac{\tau}{\lambda(t)} \right), \frac{1}{\lambda(t)^{5/2-s_p}} \partial_t u \left(\frac{x}{\lambda(t)}, t + \frac{\tau}{\lambda(t)} \right) \right) : \tau \in [0, d], t \in I \right\}$$

is uniformly bounded (and precompact) in the space $\dot{H}^s \times \dot{H}^{s-1}$. By the local theory of our equation with initial data in $\dot{H}^s \times \dot{H}^{s-1}$ (If $s = s_p$, please see proposition 3.4 and the long-time perturbation theory, otherwise see theorem 2.7 and 2.8), we have

$$\left\{ \frac{1}{\lambda(t)^{3/2-s_p}} u \left(\frac{x}{\lambda(t)}, t + \frac{\tau}{\lambda(t)} \right), \tau \in [0, d] : t \in I \right\}$$

is precompact in the space $Y_s([0, d])$. Thus we have

$$\left\| \frac{1}{\lambda(t)^{3/2-s_p}} u \left(\frac{x}{\lambda(t)}, t + \frac{\tau}{\lambda(t)} \right) \right\|_{Y_s([0, d])} \lesssim 1,$$

and

$$\left\| P_{>A} \frac{1}{\lambda(t)^{3/2-s_p}} u \left(\frac{x}{\lambda(t)}, t + \frac{\tau}{\lambda(t)} \right) \right\|_{Y_s([0, d])} \Rightarrow 0$$

as $A \rightarrow \infty$. If we rescale the first inequality back, we obtain

$$(\lambda(t))^{s_p-s} \|u\|_{Y_s([t, t+d\lambda^{-1}(t)])} \lesssim 1 \Rightarrow (\lambda(t))^{s_p-s} \|F(u)\|_{Z_s([t, t+d\lambda^{-1}(t)])} \lesssim 1,$$

which implies that $S(A)$ and $N(A)$ are uniformly bounded. In the similar way we can show $S(A)$ converges to zero as $A \rightarrow \infty$, using the uniform convergence above.

Lemma 5.1. Bilinear Estimate Suppose u_i satisfies the following linear wave equation on the time interval $I = [0, T]$, $i = 1, 2$,

$$\partial_t^2 u_i - \Delta u_i = F_i(x, t),$$

with the initial data $(u_i|_{t=0}, \partial_t u_i|_{t=0}) = (u_{0,i}, u_{1,i})$. Then

$$\begin{aligned} S &= \|(P_{>R} u_1)(P_{<r} u_2)\|_{L^{\frac{p}{s+1-(2p-2)(s-sp)}} L^{\frac{p}{2-s}}(I \times \mathbb{R}^3)} \\ &\lesssim \left(\frac{r}{R}\right)^\sigma (\|(u_{0,1}, u_{1,1})\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|F_1\|_{Z_s(I)}) \times (\|(u_{0,2}, u_{1,2})\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|F_2\|_{Z_s(I)}). \end{aligned}$$

Here the number σ is an arbitrary positive constant satisfying

$$\sigma \leq 3 \left(\frac{1}{2} - \frac{s+1-(2p-2)(s-sp)}{2p} - \frac{2-s}{2p} \right), \quad \sigma < 3 \times \frac{2-s}{2p}. \quad (21)$$

Remark We can always choose

$$\sigma = \sigma(p) = \frac{3 \min\{p-3, 1\}}{2p} > 0.$$

Proof By the Strichartz estimate

$$\begin{aligned} &\|(P_{>R} u_1)\|_{L^{\frac{2p}{s+1-(2p-2)(s-sp)}} L^{1/(\frac{2-s}{2p} + \frac{\sigma}{3})}} \\ &\lesssim \|(D_x^{-\sigma} P_{>R} u_{0,1}, D_x^{-\sigma} P_{>R} u_{1,1})\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|D_x^{-\sigma} P_{>R} F_1\|_{Z_s(I)}. \end{aligned}$$

$$\begin{aligned} &\|(P_{<r} u_2)\|_{L^{\frac{2p}{s+1-(2p-2)(s-sp)}} L^{1/(\frac{2-s}{2p} - \frac{\sigma}{3})}} \\ &\lesssim \|(D_x^\sigma P_{<r} u_{0,2}, D_x^\sigma P_{<r} u_{1,2})\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|D_x^\sigma P_{<r} F_2\|_{Z_s(I)}. \end{aligned}$$

Our choice of σ makes sure that the pairs above are admissible. Thus we have

$$\begin{aligned} &\|(P_{>R} u_1)(P_{<r} u_2)\|_{L^{\frac{p}{s+1-(2p-2)(s-sp)}} L^{\frac{p}{2-s}}} \\ &\lesssim \|(P_{>R} u_1)\|_{L^{\frac{2p}{s+1-(2p-2)(s-sp)}} L^{1/(\frac{2-s}{2p} + \frac{\sigma}{3})}} \|(P_{<r} u_2)\|_{L^{\frac{2p}{s+1-(2p-2)(s-sp)}} L^{1/(\frac{2-s}{2p} - \frac{\sigma}{3})}} \\ &\lesssim (\|(D_x^{-\sigma} P_{>R} u_{0,1}, D_x^{-\sigma} P_{>R} u_{1,1})\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|D_x^{-\sigma} P_{>R} F_1\|_{Z_s(I)}) \\ &\quad \times (\|(D_x^\sigma P_{<r} u_{0,2}, D_x^\sigma P_{<r} u_{1,2})\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|D_x^\sigma P_{<r} F_2\|_{Z_s(I)}) \\ &\lesssim \left(\frac{1}{R}\right)^\sigma (\|(P_{>R} u_{0,1}, P_{>R} u_{1,1})\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|P_{>R} F_1\|_{Z_s(I)}) \\ &\quad \times r^\sigma (\|(P_{<r} u_{0,2}, P_{<r} u_{1,2})\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|P_{<r} F_2\|_{Z_s(I)}) \\ &\lesssim \text{the right hand.} \end{aligned}$$

Lemma 5.2. Let $u(x, t)$ be a function defined on $I \times \mathbb{R}^3$, such that \hat{u} is supported in the ball $B(0, r)$ for each $t \in I$, then

$$\|P_{>R} F(u(x, t))\|_{L^{\frac{2}{s+1-(2p-2)(s-sp)}} L^{\frac{2}{2-s}}(I \times \mathbb{R}^3)} \lesssim \left(\frac{r}{R}\right)^2 \|u\|_{Y_s(I)}^p.$$

Proof

$$\begin{aligned}
& \|P_{>R}F(u(x, t))\|_{L^{\frac{2}{s+1-(2p-2)(s-sp)}} L^{\frac{2}{2-s}}(I \times \mathbb{R}^3)} \\
& \lesssim \frac{1}{R^2} \|P_{>R}\Delta_x F(u(x, t))\|_{L^{\frac{2}{s+1-(2p-2)(s-sp)}} L^{\frac{2}{2-s}}(I \times \mathbb{R}^3)} \\
& \lesssim \frac{1}{R^2} \|\Delta_x F(u(x, t))\|_{L^{\frac{2}{s+1-(2p-2)(s-sp)}} L^{\frac{2}{2-s}}(I \times \mathbb{R}^3)} \\
& \lesssim \frac{1}{R^2} \|p(\Delta_x u)|u|^{p-1} + p(p-1)|\nabla_x u|^2|u|^{p-3}u\|_{L^{\frac{2}{s+1-(2p-2)(s-sp)}} L^{\frac{2}{2-s}}} \\
& \lesssim \frac{1}{R^2} \left(\|\Delta_x u\|_{Y_s(I)} \|u\|_{Y_s(I)}^{p-1} + \|\nabla_x u\|_{Y_s(I)}^2 \|u\|_{Y_s(I)}^{p-2} \right) \\
& \lesssim \frac{r^2}{R^2} \|u\|_{Y_s(I)}^p.
\end{aligned}$$

Lemma 5.3. *Let $v(t)$ be a long-time contribution in the Duhamel formula as below*

$$v(t_0) = \int_{T_1}^{T_2} \frac{\sin((t-t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(t)) dt,$$

then for any $t_0 < T_1 < T_2$, we have

$$\|v(t_0)\|_{L^\infty(\mathbb{R}^3)} \lesssim (T_1 - t_0)^{-2/(p-1)}.$$

Proof Using the explicit expression of the wave kernel in dimension 3, we obtain

$$\begin{aligned}
& \left| \left(\int_{T_1}^{T_2} \frac{\sin((t-t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(t)) dt \right) (x) \right| \\
& = \left| \int_{T_1}^{T_2} \int_{|y-x|=t-t_0} \frac{1}{4\pi(t-t_0)} F(u(y, t)) dS(y) dt \right| \\
& \lesssim \int_{T_1}^{T_2} \int_{|y-x|=t-t_0} \frac{1}{4\pi(t-t_0)} |u(y, t)|^p dS(y) dt \\
& \lesssim \int_{T_1}^{T_2} \int_{|y-x|=t-t_0} \frac{1}{(t-t_0)} \frac{1}{|y|^{\frac{2p}{p-1}}} dS(y) dt.
\end{aligned}$$

In the last step, we use the estimate (9) for radial \dot{H}^{s_p} functions. If $|x| \leq \frac{1}{2}(T_1 - t_0)$, then on the sphere for the integral we have

$$|y| \geq |t - t_0| - |x| \geq \frac{1}{2}(t - t_0).$$

Thus for these small x ,

$$\begin{aligned}
& \left| \left(\int_{T_1}^{T_2} \frac{\sin((t-t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(t)) dt \right) (x) \right| \\
& \lesssim \int_{T_1}^{T_2} \int_{|y-x|=t-t_0} \frac{1}{(t-t_0)} \frac{1}{(t-t_0)^{2p/(p-1)}} dS(y) dt
\end{aligned}$$

$$\begin{aligned}
&\lesssim \int_{T_1}^{T_2} \int_{|y-x|=t-t_0} \frac{1}{(t-t_0)^{3+2/(p-1)}} dS(y) dt \\
&\lesssim \int_{T_1}^{T_2} \frac{(t-t_0)^2}{(t-t_0)^{3+2/(p-1)}} dt \\
&\lesssim \int_{T_1}^{T_2} \frac{1}{(t-t_0)^{1+2/(p-1)}} dt \\
&\lesssim (T_1 - t_0)^{-2/(p-1)}.
\end{aligned}$$

On the other hand, if $x \geq \frac{1}{2}(T_1 - t_0)$, by (7) we have

$$\begin{aligned}
&\left| \left(\int_{T_1}^{T_2} \frac{\sin((t-t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(t)) dt \right) (x) \right| \\
&\lesssim \frac{1}{|x|^{2/(p-1)}} \left\| \int_{T_1}^{T_2} \frac{\sin((t-t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(t)) dt \right\|_{\dot{H}^{s_p}} \\
&\lesssim \frac{1}{(T_1 - t_0)^{2/(p-1)}}.
\end{aligned}$$

Combining these two cases, we finish our proof.

Lemma 5.4. *Suppose $S(A)$ is a nonnegative function defined in \mathbb{R}^+ satisfying $S(A) \rightarrow 0$ as $A \rightarrow \infty$. In addition, there exist $\alpha, \beta \in (0, 1)$ and $l, \omega > 0$ with*

$$l\alpha + \beta > 1,$$

such that

$$S(A) \lesssim S(A^\beta) S^l(A^\alpha) + A^{-\omega} \quad (22)$$

is true for each sufficiently large A . Then

$$S(A) \lesssim A^{-\omega}$$

for each sufficiently large A .

Proof Let us first choose two constants l^- and ω^- , which are slightly smaller than l and ω respectively, such that the inequality $l^-\alpha + \beta > 1$ still holds. By the conditions given, we can find a constant $A_0 \gg 1$, such that the following inequalities hold

$$S(A) \leq \frac{1}{2} S(A^\beta) S^{l^-}(A^\alpha) + \frac{1}{2} A^{-\omega^-} \text{ if } A \geq A_0. \quad (23)$$

$$S(A) < 1/2 \text{ if } A \geq A_0^{1/\alpha}.$$

Using the second inequality above, we know the following inequality holds for all $A \in [A_0^\alpha, A_0]$ if ω_1 is sufficiently small

$$S(A) \leq A^{-\omega_1}. \quad (24)$$

Fix such a small constant $\omega_1 \leq \omega^-$. We will show that the inequality (24) above holds for each $A \geq A_0^\alpha$ by an induction. We already know this is true for $A \in [A_0^\alpha, A_0]$. If

$A \in [A_0, A_0^{1/\beta}]$, the inequality (23) implies

$$\begin{aligned}
S(A) &\leq \frac{1}{2}S(A^\beta)S^{l^-}(A^\alpha) + \frac{1}{2}A^{-\omega^-} \\
&\leq \frac{1}{2}(A^\beta)^{-\omega_1}((A^\alpha)^{-\omega_1})^{l^-} + \frac{1}{2}A^{-\omega^-} \\
&\leq \frac{1}{2}(A^{-\omega_1})^{\beta+l^-\alpha} + \frac{1}{2}A^{-\omega_1} \\
&\leq A^{-\omega_1}.
\end{aligned}$$

Here we use the fact that $A^\alpha, A^\beta \in [A_0^\alpha, A_0]$ if A satisfies our assumption. Conducting an induction, we can show the inequality holds for each $A \in [A_0^{(1/\beta)^n}, A_0^{(1/\beta)^{n+1}}]$ if n is a nonnegative integer. In summary, the inequality (24) is true for each $A \geq A_0^\alpha$. Plugging this back in the original recurrence formula (22), we obtain for sufficiently large A ,

$$S(A) \lesssim A^{-\omega_1(\beta+l\alpha)} + A^{-\omega} \lesssim A^{-\min\{\omega_1(\beta+l\alpha), \omega\}},$$

which indicates faster decay than $A^{-\omega_1}$. Iterating the argument if necessary, we gain the decay $S(A) \lesssim A^{-\omega}$ and finish the proof.

5.2 Recurrence Formula

Under our setting in this section, given $0 < \alpha < \beta < 1$ and $\varepsilon_1 > 0$, we have the following recurrence formula for sufficiently large A

$$N(A) \lesssim S(A^\beta)S^{p-1}(A^\alpha) + A^{-(\beta-\alpha)\sigma(p)} + A^{-2(1-\beta)}; \quad (25)$$

$$S(A) \lesssim N(A^{1-\varepsilon_1}) + A^{-\sigma_1(p)}. \quad (26)$$

The constants $\sigma(p), \sigma_1(p)$ depend on p but nothing else.

Proof of the first inequality In the following argument, all the space-time norms are taken in $[t, t + d\lambda^{-1}(t)] \times \mathbb{R}^3$.

$$\begin{aligned}
&\|P_{>\lambda(t)A}(F(u))\|_{Z_s} \\
&\lesssim \lambda(t)^{-(p-1)(s-s_p)} \|P_{>\lambda(t)A}F(u)\|_{L^{\frac{2}{s+1-(2p-2)(s-s_p)}} L^{\frac{2}{2-s}}} \\
&\leq \lambda(t)^{-(p-1)(s-s_p)} \|P_{>\lambda(t)A}(F(u_{\leq A^\beta \lambda(t)}))\|_{L^{\frac{2}{s+1-(2p-2)(s-s_p)}} L^{\frac{2}{2-s}}} \\
&+ \lambda(t)^{-(p-1)(s-s_p)} \|P_{>\lambda(t)A}(F(u) - F(u_{\leq A^\beta \lambda(t)}))\|_{L^{\frac{2}{s+1-(2p-2)(s-s_p)}} L^{\frac{2}{2-s}}} \\
&\leq \lambda(t)^{-(p-1)(s-s_p)} (I_1 + I_2).
\end{aligned}$$

By lemma 5.2, we have

$$I_1 \lesssim \left(\frac{A^\beta}{A}\right)^2 \|u\|_{Y_s}^p \lesssim (\lambda(t))^{p(s-s_p)} A^{-2(1-\beta)}.$$

For I_2 , we have (all norms unmarked are $L^{\frac{2}{s+1-(2p-2)(s-sp)}} L^{\frac{2}{2-s}}([t, t+d\lambda^{-1}(t)] \times \mathbb{R}^3)$ norms)

$$\begin{aligned}
I_2 &\leq \left\| P_{>\lambda(t)A} \left[u_{>A^\beta\lambda(t)} \int_0^1 F'(u_{\leq A^\beta\lambda(t)} + \tau u_{>A^\beta\lambda(t)}) d\tau \right] \right\| \\
&\lesssim \left\| u_{>A^\beta\lambda(t)} \int_0^1 F'(u_{\leq A^\beta\lambda(t)} + \tau u_{>A^\beta\lambda(t)}) d\tau \right\| \\
&\lesssim \left\| \begin{aligned} &u_{>A^\beta\lambda(t)} \int_0^1 F'(u_{\leq A^\beta\lambda(t)} + \tau u_{>A^\beta\lambda(t)}) d\tau \\ &- u_{>A^\beta\lambda(t)} \int_0^1 F'(u_{A^\alpha\lambda(t) < \cdot \leq A^\beta\lambda(t)} + \tau u_{>A^\beta\lambda(t)}) d\tau \end{aligned} \right\| \\
&\quad + \left\| u_{>A^\beta\lambda(t)} \int_0^1 F'(u_{A^\alpha\lambda(t) < \cdot \leq A^\beta\lambda(t)} + \tau u_{>A^\beta\lambda(t)}) d\tau \right\| \\
&\lesssim \left\| u_{>A^\beta\lambda(t)} u_{\leq A^\alpha\lambda(t)} \int_0^1 \int_0^1 F''(\tilde{\tau} u_{\leq A^\alpha\lambda(t)} + u_{A^\alpha\lambda(t) < \cdot \leq A^\beta\lambda(t)} + \tau u_{>A^\beta\lambda(t)}) d\tau d\tilde{\tau} \right\| \\
&\quad + \left\| u_{>A^\beta\lambda(t)} \int_0^1 F'(u_{A^\alpha\lambda(t) < \cdot \leq A^\beta\lambda(t)} + \tau u_{>A^\beta\lambda(t)}) d\tau \right\| \\
&\lesssim \left\| u_{>A^\beta\lambda(t)} u_{\leq A^\alpha\lambda(t)} \right\|_{L^{\frac{p}{s+1-(2p-2)(s-sp)}} L^{\frac{p}{2-s}}} \\
&\quad \times \left\| \int_0^1 \int_0^1 F''(\tilde{\tau} u_{\leq A^\alpha\lambda(t)} + u_{A^\alpha\lambda(t) < \cdot \leq A^\beta\lambda(t)} + \tau u_{>A^\beta\lambda(t)}) d\tau d\tilde{\tau} \right\|_{L^{\frac{2p}{(p-2)(s+1-(2p-2)(s-sp))}} L^{\frac{2p}{(p-2)(2-s)}}} \\
&\quad + \left\| u_{>A^\beta\lambda(t)} \right\|_{L^{\frac{2p}{s+1-(2p-2)(s-sp)}} L^{\frac{2p}{2-s}}} \\
&\quad \times \left\| \int_0^1 F'(u_{A^\alpha\lambda(t) < \cdot \leq A^\beta\lambda(t)} + \tau u_{>A^\beta\lambda(t)}) d\tau \right\|_{L^{\frac{2p}{(p-1)(s+1-(2p-2)(s-sp))}} L^{\frac{2p}{(p-1)(2-s)}}} \\
&\lesssim (\lambda(t))^{p(s-sp)} \left[\left(\frac{A^\alpha\lambda(t)}{A^\beta\lambda(t)} \right)^{\sigma(p)} + S(A^\beta) S^{p-1}(A^\alpha) \right] \\
&\lesssim (\lambda(t))^{p(s-sp)} \left(A^{-(\beta-\alpha)\sigma(p)} + S(A^\beta) S^{p-1}(A^\alpha) \right).
\end{aligned}$$

The bilinear estimate is used here to estimate the term $u_{>A^\beta\lambda(t)} u_{\leq A^\alpha\lambda(t)}$. Collecting both terms and taking sup for all $t \in I$, we obtain the first inequality.

Proof of the second inequality To prove the inequality (26) we first define t_i for $i \geq 1$ given $t_0 \in I$.

$$t_i = t_{i-1} + d\lambda^{-1}(t_{i-1}). \quad (27)$$

By the choice of d (please see proposition 3.3), all t_i 's are in the maximal lifespan I . By the Strichartz estimate and the Duhamel formula, we have

$$\begin{aligned}
&\|u_{>\lambda(t_0)A}\|_{Y_s([t_0, t_1])} \\
&= \left\| \int_t^\infty \frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} P_{>\lambda(t_0)A} F(u(\tau)) d\tau \right\|_{Y_s([t_0, t_1])} \\
&\leq \left\| \int_t^{t_2} \frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} P_{>\lambda(t_0)A} F(u(\tau)) d\tau \right\|_{Y_s([t_0, t_1])}
\end{aligned}$$

$$\begin{aligned}
& + \liminf_{T \rightarrow \infty} \left\| \int_{t_2}^T \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} P_{>\lambda(t_0)A} F(u(\tau)) d\tau \right\|_{Y_s([t_0, t_1])} \\
& \lesssim \|P_{>\lambda(t_0)A} F(u)\|_{Z_s([t_0, t_2] \times \mathbb{R}^3)} \\
& + \liminf_{T \rightarrow \infty} \left\| \int_{t_2}^T \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} P_{>\lambda(t_0)A} F(u(\tau)) d\tau \right\|_{Y_s([t_0, t_1])} \\
& = I_1 + I_2.
\end{aligned}$$

The first term can be dominated by

$$\begin{aligned}
I_1 & \lesssim \|P_{>\lambda(t_0)A} F(u)\|_{Z_s([t_0, t_1] \times \mathbb{R}^3)} + \|P_{>\lambda(t_0)A} F(u)\|_{Z_s([t_1, t_2] \times \mathbb{R}^3)} \\
& \lesssim (\lambda(t_0))^{s-s_p} N(A) + (\lambda(t_1))^{s-s_p} N\left(\frac{\lambda(t_0)}{\lambda(t_1)} A\right) \\
& \lesssim (\lambda(t_0))^{s-s_p} N(A^{1-\varepsilon_1}).
\end{aligned}$$

for any small positive number ε_1 and sufficiently large $A > A_0(u, \varepsilon_1)$, because $\lambda(t_0)$ and $\lambda(t_1)$ are comparable to each other by the local compactness result (11).

Now let us consider the term I_2 . First of all, by (7), we have

$$\begin{aligned}
& \left\| \int_{t_2}^T \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} P_{>\lambda(t_0)A} F(u(\tau)) d\tau \right\|_{L^\infty L^2([t_0, t_1] \times \mathbb{R}^3)} \\
& \lesssim \frac{1}{(\lambda(t_0)A)^{s_p}} \left\| \int_{t_2}^T \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau)) d\tau \right\|_{L_{[t_0, t_1]}^\infty \dot{H}^{s_p}(\mathbb{R}^3)} \\
& \lesssim \frac{1}{(\lambda(t_0)A)^{s_p}}
\end{aligned}$$

Using lemma 5.3, we also obtain

$$\begin{aligned}
& \left\| \int_{t_2}^T \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} P_{>\lambda(t_0)A} F(u(\tau)) d\tau \right\|_{L^\infty L^\infty([t_0, t_1] \times \mathbb{R}^3)} \\
& \lesssim \left\| \int_{t_2}^T \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau)) d\tau \right\|_{L^\infty L^\infty([t_0, t_1] \times \mathbb{R}^3)} \\
& \lesssim (t_2 - t_1)^{-2/(p-1)} \\
& \lesssim (\lambda(t_0))^{2/(p-1)}.
\end{aligned}$$

Using the interpolation between L^2 and L^∞ , we have

$$\begin{aligned}
& \left\| P_{>\lambda(t_0)A} \int_{t_2}^T \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau)) d\tau \right\|_{L^\infty L^{\frac{6}{3-2s}}([t_0, t_1] \times \mathbb{R}^3)} \\
& \leq \|\cdot\|_{L^\infty L^\infty([t_0, t_1] \times \mathbb{R}^3)}^{2s/3} \|\cdot\|_{L^\infty L^2([t_0, t_1] \times \mathbb{R}^3)}^{(3-2s)/3} \\
& \lesssim [\lambda(t_0)^{2/(p-1)}]^{2s/3} [(\lambda(t_0)A)^{-s_p}]^{(3-2s)/3} \\
& = (\lambda(t_0))^{s-s_p} A^{\frac{-s_p(3-2s)}{3}}.
\end{aligned}$$

Next we will use the interpolation again to gain the estimate of Y_s norm. There are two technical lemmas.

Lemma 5.5. *There exists a constant $\kappa = \kappa(p) \in (0, 1)$ that depends only on p , so that for each $s \in [s_p, 1)$, there exists an s -admissible pair (q, r) , with $q \neq \infty$ and*

$$\frac{s+1-(2p-2)(s-s_p)}{2p} = \kappa \cdot 0 + (1-\kappa)\frac{1}{q}; \quad \frac{2-s}{2p} = \kappa\frac{3-2s}{6} + (1-\kappa)\frac{1}{r}.$$

Proof This is just a basic and boring computation. Please see the Appendix.

Lemma 5.6. *Given any s -admissible pair (q, r) with $q < \infty$, we have*

$$\lim_{T \rightarrow \infty} \left\| \int_{t_2}^T \frac{\sin((\tau - t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau)) d\tau \right\|_{L^q L^r([t_0, t_1] \times \mathbb{R}^3)} \leq C(\lambda(t_0))^{s-s_p}.$$

The constant C does not depend on t_0 .

Proof By lemma 10.4, we have

$$\lim_{T \rightarrow \infty} \int_{t_2}^T \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau)) d\tau = S(t - t_2)(u(t_2), \partial_t u(t_2))$$

in the space $L^q L^r([t_0, t_1] \times \mathbb{R}^3)$. Thus

$$\begin{aligned} & \lim_{T \rightarrow \infty} \left\| \int_{t_2}^{T_i} \frac{\sin((\tau - t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}} (u(\tau)) d\tau \right\|_{L^q L^r([t_0, t_1])} \\ &= \|S(t - t_2)(u(t_2), \partial_t u(t_2))\|_{L^q L^r([t_0, t_1] \times \mathbb{R}^3)} \\ &\lesssim \|(u(t_2), \partial_t u(t_2))\|_{\dot{H}^s \times \dot{H}^{s-1}} \\ &\lesssim (\lambda(t_2))^{s-s_p} \\ &\lesssim (\lambda(t_0))^{s-s_p}. \end{aligned}$$

Now let us apply the two lemmas

$$\begin{aligned} I_2 &= \liminf_{T \rightarrow \infty} \left\| \int_{t_2}^T \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} P_{>\lambda(t_0)A} F(u(\tau)) d\tau \right\|_{Y_s([t_0, t_1])} \\ &\leq \liminf_{T \rightarrow \infty} \left(\left\| \int_{t_2}^T \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} P_{>\lambda(t_0)A} F(u(\tau)) d\tau \right\|_{L^\infty L^{\frac{6}{3-2s}}([t_0, t_1] \times \mathbb{R}^3)}^{\kappa(p)} \right. \\ &\quad \left. \times \left\| \int_{t_2}^T \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} P_{>\lambda(t_0)A} F(u(\tau)) d\tau \right\|_{L^q L^r([t_0, t_1] \times \mathbb{R}^3)}^{1-\kappa(p)} \right) \\ &\lesssim \left[(\lambda(t_0))^{s-s_p} A^{\frac{-s_p(3-2s)}{3}} \right]^{\kappa(p)} \times \lim_{T \rightarrow \infty} \left\| \int_{t_2}^T \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau)) d\tau \right\|_{L^q L^r}^{1-\kappa(p)} \\ &\lesssim \left[(\lambda(t_0))^{s-s_p} A^{\frac{-s_p(3-2s)}{3}} \right]^{\kappa(p)} (\lambda(t_0))^{(s-s_p)(1-\kappa(p))} \\ &\lesssim (\lambda(t_0))^{s-s_p} A^{\frac{-s_p \kappa(p)(3-2s)}{3}} \\ &\lesssim (\lambda(t_0))^{s-s_p} A^{-\sigma_1(p)} \end{aligned}$$

Here $\sigma_1(p) = \kappa(p)/6$. It depends only on p .

Combining I_1 and I_2 and then taking the sup for all $t \in I$, we finish the proof of the second inequality.

5.3 Decay of $S(A)$ and $N(A)$

Plugging the first recurrence formula into the second one, we gain

$$\begin{aligned} S(A) \lesssim S(A^{(1-\varepsilon_1)\beta}) S^{p-1}(A^{(1-\varepsilon_1)\alpha}) &+ A^{-\sigma(p)(1-\varepsilon_1)(\beta-\alpha)} \\ &+ A^{-2(1-\varepsilon_1)(1-\beta)} + A^{-\sigma_1(p)}. \end{aligned}$$

Choose α , β and ε_1 so that

$$(1 - \varepsilon_1)\beta = 2/3; (1 - \varepsilon_1)\alpha = 1/3; \varepsilon_1 = 1/10000. \quad (28)$$

Then we have

$$S(A) \lesssim S(A^{2/3}) S^{p-1}(A^{1/3}) + A^{-\sigma_2(p)}.$$

for sufficiently large A . Here the positive number $\sigma_2(p)$ depends on p only.

$$\sigma_2 = \min\{\sigma(p)/3, \sigma_1(p), 0.6\}.$$

Applying lemma 5.4, we have $S(A) \lesssim A^{-\sigma_2(p)}$ for sufficiently large A . Plugging this in the first recurrence formula, we have $N(A) \lesssim A^{-\sigma_2(p)}$ for large A . Observing that both $S(A)$ and $N(A)$ is uniformly bounded, we know these two decay estimates are actually valid for each $A > 0$. Now let us choose

$$s_1 = \min\{1, s + \frac{99}{100}\sigma_2(p)\};$$

and define (local contribution of the Duhamel Formula)

$$\begin{aligned} v_{t'}(t) &= \int_{t'}^{t'+d\lambda(t')^{-1}} \frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau)) d\tau; \\ \partial_t v_{t'}(t) &= - \int_{t'}^{t'+d\lambda(t')^{-1}} \cos((\tau-t)\sqrt{-\Delta}) F(u(\tau)) d\tau. \end{aligned}$$

We obtain for any $t \leq t'$ and integer $k \geq 0$

$$\begin{aligned} &\left\| P_{\lambda(t')2^k < \cdot < \lambda(t')2^{k+1}}(v_{t'}(t), \partial_t v_{t'}(t)) \right\|_{\dot{H}^{s_1} \times \dot{H}^{s_1-1}} \\ &\lesssim (\lambda(t')2^k)^{s_1-s} \left\| P_{\lambda(t')2^k < \cdot < \lambda(t')2^{k+1}}(v_{t'}(t), \partial_t v_{t'}(t)) \right\|_{\dot{H}^s \times \dot{H}^{s-1}} \\ &\lesssim (\lambda(t')2^k)^{s_1-s} \left\| P_{>\lambda(t')2^k}(v_{t'}(t), \partial_t v_{t'}(t)) \right\|_{\dot{H}^s \times \dot{H}^{s-1}} \\ &\lesssim (\lambda(t')2^k)^{s_1-s} \left\| P_{>\lambda(t')2^k} F(u) \right\|_{Z_s([t', t'+d\lambda(t')^{-1}])} \\ &\lesssim (\lambda(t')2^k)^{s_1-s} (\lambda(t'))^{s-s_p} N(2^k) \\ &\lesssim (\lambda(t'))^{s_1-s_p} (2^k)^{s_1-s-\sigma_2(p)}. \end{aligned}$$

Summing for all $k \geq 0$, we have

$$\left\| P_{>\lambda(t')}(v_{t'}(t), \partial_t v_{t'}(t)) \right\|_{\dot{H}^{s_1} \times \dot{H}^{s_1-1}} \lesssim (\lambda(t'))^{s_1-s_p}.$$

Combining this with

$$\begin{aligned}
& \|P_{\leq \lambda(t')}(v_{t'}(t), \partial_t v_{t'}(t))\|_{\dot{H}^{s_1} \times \dot{H}^{s_1-1}} \\
& \lesssim (\lambda(t'))^{s_1-s_p} \|P_{\leq \lambda(t')}(v_{t'}(t), \partial_t v_{t'}(t))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} \\
& \lesssim (\lambda(t'))^{s_1-s_p},
\end{aligned}$$

we obtain

$$\|(v_{t'}(t), \partial_t v_{t'}(t))\|_{\dot{H}^{s_1} \times \dot{H}^{s_1-1}} \lesssim (\lambda(t'))^{s_1-s_p}. \quad (29)$$

5.4 Higher Regularity

In this section we will show $(u(x, t), \partial_t u(x, t)) \in \dot{H}^{s_1} \times \dot{H}^{s_1-1}(\mathbb{R}^3)$ for each $t \in I$.

Center estimate Let us break the Duhamel formula into two pieces.

$$\begin{aligned}
u^{(1)}(t) &= \int_t^{t_1} \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau)) d\tau; \\
u^{(2)}(t) &= \int_{t_1}^\infty \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau)) d\tau.
\end{aligned}$$

Let χ be the characteristic function of the region $\{(x, t) : |x| > \frac{d\lambda^{-1}(t_0)}{2} + |t - t_1|\}$. We have

$$\begin{aligned}
& \|\chi F(u(t))\|_{L^1 L^{\frac{6}{5-2s_1}}([t_1, \infty) \times \mathbb{R}^3)} \\
&= \int_{t_1}^\infty \left(\int_{|x| > \frac{d\lambda^{-1}(t_0)}{2} + |t-t_1|} (F(u))^{\frac{6}{5-2s_1}} dx \right)^{\frac{5-2s_1}{6}} dt \\
&\lesssim \int_{t_1}^\infty \left(\int_{|x| > \frac{d\lambda^{-1}(t_0)}{2} + |t-t_1|} \left(\frac{1}{|x|^{\frac{2p}{p-1}}} \right)^{\frac{6}{5-2s_1}} dx \right)^{\frac{5-2s_1}{6}} dt \\
&\lesssim \int_{t_1}^\infty \left(\frac{1}{\left| \frac{d\lambda^{-1}(t_0)}{2} + t - t_1 \right|^{\frac{2p}{p-1} \frac{6}{5-2s_1} - 3}} \right)^{\frac{5-2s_1}{6}} dt \\
&\lesssim \int_{t_1}^\infty \left(\frac{d\lambda^{-1}(t_0)}{2} + t - t_1 \right)^{s_p - s_1 - 1} dt \\
&\lesssim \lambda(t_0)^{s_1-s_p}.
\end{aligned}$$

By lemma 2.12, there exists a pair $(\tilde{u}_0, \tilde{u}_1)$ so that

$$\begin{aligned}
& \|(\tilde{u}_0, \tilde{u}_1)\|_{\dot{H}^{s_1} \times \dot{H}^{s_1-1}(\mathbb{R}^3)} \lesssim \lambda(t_0)^{s_1-s_p}, \\
& (u^{(2)}(t_0), \partial_t u^{(2)}(t_0)) = (\tilde{u}_0, \tilde{u}_1) \text{ in } B\left(0, \frac{d\lambda^{-1}(t_0)}{2}\right).
\end{aligned}$$

This implies

$$(u(t_0), \partial_t u(t_0)) = (\tilde{u}_0 + u^{(1)}(t_0), \tilde{u}_1 + \partial_t u^{(1)}(t_0)) \text{ in } B\left(0, \frac{d\lambda^{-1}(t_0)}{2}\right). \quad (30)$$

By (29), we have

$$\|(u^{(1)}(t_0), \partial_t u^{(1)}(t_0))\|_{\dot{H}^{s_1} \times \dot{H}^{s_1-1}} \lesssim \lambda(t_0)^{s_1-s_p}.$$

Combining this with the $\dot{H}^{s_1} \times \dot{H}^{s_1-1}$ bound of $(\tilde{u}_0, \tilde{u}_1)$, we have

$$\|(\tilde{u}_0 + u^{(1)}(t_0), \tilde{u}_1 + \partial_t u^{(1)}(t_0))\|_{\dot{H}^{s_1} \times \dot{H}^{s_1-1}} \lesssim \lambda(t_0)^{s_1-s_p}. \quad (31)$$

Tail Estimate Let $(u'_0, u'_1) = \Psi_{d\lambda^{-1}(t_0)/4}(u(t_0), \partial_t u(t_0))$, and

$$\frac{1}{q} = \frac{1}{2} + \frac{1-s_1}{3}.$$

By theorem 4.1, if $r \geq d\lambda^{-1}(t_0)/4$, we have

$$\begin{aligned} \left(\int_{r < |x| < 4r} (|\nabla u'_0|^q + |u'_1|^q) dx \right)^{1/q} &\lesssim \left(\int_{r < |x| < 4r} (|\nabla u'_0|^2 + |u'_1|^2) dx \right)^{1/2} (r^3)^{\frac{1}{q}-\frac{1}{2}} \\ &\lesssim r^{-(1-s_p)} (r^3)^{(1-s_1)/3} \\ &\lesssim r^{-(s_1-s_p)}. \end{aligned}$$

Letting $r = 2^k d\lambda^{-1}(t_0)/4$ and summing for all $k \geq 0$, we obtain that the pair (u'_0, u'_1) is in the space $\dot{W}^{1,q} \times L^q(\mathbb{R}^3)$ with

$$\|(u'_0, u'_1)\|_{\dot{W}^{1,q} \times L^q(\mathbb{R}^3)} \lesssim (d\lambda(t_0)^{-1}/4)^{-(s_1-s_p)} \lesssim (\lambda(t_0))^{s_1-s_p}.$$

By the Sobolev embedding, we have

$$\|(u'_0, u'_1)\|_{\dot{H}^{s_1} \times \dot{H}^{s_1-1}(\mathbb{R}^3)} \lesssim (\lambda(t_0))^{s_1-s_p}. \quad (32)$$

Considering (30), (31), (32) and using lemma 2.9, we have

$$\|(u(t_0), \partial_t u(t_0))\|_{\dot{H}^{s_1} \times \dot{H}^{s_1-1}(\mathbb{R}^3)} \lesssim (\lambda(t_0))^{s_1-s_p}.$$

5.5 Conclusion

There are two cases

Case 1 ($s_1 = 1$) Now we have finished our argument and obtain the energy estimate.

Case 2 ($s_1 < 1$) This means $s_1 = s + 0.99\sigma_2(p)$. Now let us consider the set

$$\left\{ \left(\frac{1}{\lambda(t)^{3/2-s_p}} u\left(\frac{x}{\lambda(t)}, t\right), \frac{1}{\lambda(t)^{5/2-s_p}} \partial_t u\left(\frac{x}{\lambda(t)}, t\right) \right) : t \in I \right\}$$

This is precompact in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$, and bounded in the space $\dot{H}^{s+0.99\sigma_2(p)} \times \dot{H}^{s-1+0.99\sigma_2(p)}$, thus it is also precompact in the space $\dot{H}^{s+0.98\sigma_2(p)} \times \dot{H}^{s-1+0.98\sigma_2(p)}$.

6 Global Energy Estimate and its Corollary

Repeat the recurrence process we described in the previous section starting from the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$. Each time we either gain the global energy estimate below or gain additional regularity by $0.98\sigma_2(p)$. This number depends on p only. As a result, the process has to stop at $\dot{H}^1 \times L^2$ after finite steps.

Proposition 6.1. Global Energy Estimate *Let $u(x, t)$ be a minimal blow-up solution. Then $(u(t_0), \partial_t u(t_0))$ is in the energy space for each $t_0 \in I$ with*

$$\|(u(t_0), \partial_t u(t_0))\|_{\dot{H}^1 \times L^2(\mathbb{R}^3)} \lesssim \lambda(t_0)^{1-s_p}. \quad (33)$$

By the local theory, we actually obtain

$$(u(t), \partial_t u(t)) \in C(I; \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)).$$

Remark By lemma 4.2, we have the following holds for any $0 < a < b < \infty$

$$(\partial_r w(t), \partial_t w(t)) \in C(I; L^2 \times L^2([a, b]))$$

6.1 Self-similar and High-to-low Frequency Cascade Cases

In both two cases, we can choose $t_i \rightarrow \infty$ such that $\lambda(t_i) \rightarrow 0$. This implies

$$\int_{\mathbb{R}^3} (|\nabla u(t_i)|^2 + |\partial_t u(t_i)|^2) dx \rightarrow 0.$$

By the Sobolev embedding, we have

$$\|u\|_{L^{p+1}(\mathbb{R}^3)}^{p+1} \leq \|u\|_{L^{\frac{3}{2}(p-1)}(\mathbb{R}^3)}^{p-1} \|u\|_{L^6(\mathbb{R}^3)}^2 \lesssim \|u\|_{\dot{H}^{s_p}(\mathbb{R}^3)}^{p-1} \|u\|_{\dot{H}^1(\mathbb{R}^3)}^2. \quad (34)$$

This implies $\|u(t_i)\|_{L^{p+1}(\mathbb{R}^3)}^{p+1} \rightarrow 0$. Using the definition of energy we have $E(t_i) \rightarrow 0$. On the other hand, we know the energy is a constant. Therefore the energy must be zero.

Defocusing Case It is nothing to say, because in this case an energy zero means that the solution is identically zero.

Focusing Case We can still solve the problem using the following theorem. By the fact that the energy is zero, we know u blows up in finite time in both time directions. But this is a contradiction with our assumption $T_+ = \infty$.

Theorem 6.2. *(Please see theorem 3.1 in [13], Non-positive energy implies blowup)*
Let $(u_0, u_1) \in (\dot{H}^1 \times L^2) \cap (\dot{H}^{s_p} \times \dot{H}^{s_p-1})$ be initial data. Assume that (u_0, u_1) is not identically zero and satisfies $E(u_0, u_1) \leq 0$. Then the maximal life-span solution to the non-linear wave equation blows up both forward and backward in finite time.

6.2 Soliton-like Solutions in the Defocusing Case

Now let us consider the soliton-like solutions in the defocusing case. First we have a useful global integral estimate in the defocusing case.

Lemma 6.3. *(Please see [18]) Let u be a solution of (1) defined in a time interval $[0, T]$ with $(u, \partial_t u) \in \dot{H}^1 \times L^2$ and a finite energy*

$$E = \int_{\mathbb{R}^3} \left(\frac{1}{2} |\nabla_x u|^2 + \frac{1}{2} |\partial_t u|^2 + \frac{1}{p+1} |u(x)|^{p+1} \right) dx.$$

For any $R > 0$, we have

$$\begin{aligned} & \frac{1}{2R} \int_0^T \int_{|x| < R} (|\nabla u|^2 + |\partial_t u|^2) dx dt + \frac{1}{2R^2} \int_0^T \int_{|x|=R} |u|^2 d\sigma_R dt \\ & + \frac{1}{2R} \frac{2p-4}{p+1} \int_0^T \int_{|x| < R} |u|^{p+1} dx dt + \frac{p-1}{p+1} \int_0^T \int_{|x| > R} \frac{|u|^{p+1}}{|x|} dx dt \\ & + \frac{2}{R^2} \int_{|x| < R} |u(T)|^2 dx \leq 2E. \end{aligned}$$

Observing that each term on the left hand is nonnegative, we can obtain a uniform upper bound for the last term in the second line above

$$\int_0^T \int_{|x| > R} \frac{|u|^{p+1}}{|x|} dx dt \leq \frac{2(p+1)}{p-1} E.$$

Letting R approach zero and T approach T_+ , we have

$$\int_0^{T_+} \int_{\mathbb{R}^3} \frac{|u|^{p+1}}{|x|} dx dt \leq \frac{2(p+1)}{p-1} E. \quad (35)$$

The energy E here is finite by our estimate (34). On the other hand, recalling our local compactness result (See lemma 3.5), we obtain $(T_+ = \infty)$

$$\int_0^\infty \int_{\mathbb{R}^3} \frac{|u|^{p+1}}{|x|} dx dt = \infty.$$

This finishes our discussion in this case.

7 Further Estimates in the Soliton-like Case

Let u be a soliton-like minimal blow-up solution. We will find additional decay of u at infinity. The method used here is similar to the one C.E.Kenig and F.Merle used in their paper [11] for super-critical case. Throughout this section $w(r, t)$, $h(r, t)$, $z_1(r, t)$ and $z_2(r, t)$ are defined as usual using $u(x, t)$.

Remark The argument in this section works in both the defocusing and focusing case. But we are particularly interested in the focusing case, because the soliton-like solutions in the focusing case are the only solutions that still survive.

7.1 Setup

Let $\varphi(x)$ be a smooth cutoff function in \mathbb{R}^3 .

$$\varphi(x) \begin{cases} = 0, & |x| \leq 1/2; \\ \in [0, 1], & 1/2 \leq |x| \leq 1; \\ = 1, & |x| \geq 1. \end{cases}$$

Then by the compactness of u (Please see lemma 3.4), $\|\varphi(x/R)u(x, t)\|_{\dot{H}^{s_p}}$ converges to zero uniformly in t as $R \rightarrow \infty$. Thus we have a positive function $g(r)$ so that $g(r)$ decreases to zero as r increases to infinity with

$$\|\varphi(x/R)u(x, t)\|_{\dot{H}^{s_p}} \leq g(R).$$

This means for each $|x| \geq R$

$$|u(x, t)| = |\varphi(x/R)u(x, t)| \leq C \frac{\|\varphi(\cdot/R)u(\cdot, t)\|_{\dot{H}^{s_p}}}{|x|^{2/(p-1)}} \leq \frac{Cg(R)}{|x|^{2/(p-1)}}.$$

Let us define

$$f_\beta(r) = \sup_{t \in \mathbb{R}, |x| \geq r} |x|^\beta |u(x, t)|$$

for $\beta \in [2/(p-1), 1)$ and $r > 0$. This is a nonincreasing function of r defined from \mathbb{R}^+ to $[0, \infty) \cup \{\infty\}$. Consider the set

$$U = \{\beta \in [2/(p-1), 1) : f_\beta(r) \rightarrow 0 \text{ as } r \rightarrow \infty\}.$$

This is not empty, since $2/(p-1)$ is in U . Due to the estimate

$$|x|^\beta |u(x, t)| \leq C_p |x|^{\beta - \frac{2}{p-1}} \|u(\cdot, t)\|_{\dot{H}^{s_p}},$$

we know if $\beta \in U$, then $f_\beta(r)$ is a bounded function. By definition of f_β , we have for any time t and $|x| \geq r$

$$|u(x, t)| \leq \frac{f_\beta(r)}{|x|^\beta}. \quad (36)$$

This is a meaningful inequality as long as $\beta \in U$.

Local Energy of w Let $\beta \in U$. Applying lemma 4.3 to w we have

$$\begin{aligned} \left(\int_{r_0}^{4r_0} |z_1(r, t_0)|^2 dr \right)^{1/2} &\leq \left(\int_{r_0+M}^{4r_0+M} |z_1(r, t_0 + M)|^2 dr \right)^{1/2} \\ &\quad + \left(\int_{r_0}^{4r_0} \left(\int_0^M h(r+t, t_0+t) dt \right)^2 dr \right)^{1/2}. \end{aligned}$$

Let $M \rightarrow \infty$. Using proposition 4.9 we have

$$\left(\int_{r_0}^{4r_0} |z_1(r, t_0)|^2 dr \right)^{1/2} \leq \limsup_{M \rightarrow \infty} \left(\int_{r_0}^{4r_0} \left(\int_0^M (r+t) F(u(r+t, t_0+t)) dt \right)^2 dr \right)^{1/2}$$

$$\begin{aligned}
&\leq \limsup_{M \rightarrow \infty} \left(\int_{r_0}^{4r_0} \left(\int_0^M (r+t) \left(\frac{f_\beta(r_0)}{(r+t)^\beta} \right)^p dt \right)^2 dr \right)^{1/2} \\
&\lesssim_p \limsup_{M \rightarrow \infty} \left(\int_{r_0}^{4r_0} \left(\frac{f_\beta^p(r_0)}{r^{p\beta-2}} \right)^2 dr \right)^{1/2} \\
&\leq f_\beta^p(r_0) \left(\int_{r_0}^{4r_0} \frac{1}{r^{2p\beta-4}} dr \right)^{1/2} \\
&\lesssim_p f_\beta^p(r_0) \left(\frac{1}{r_0^{2p\beta-5}} \right)^{1/2} \\
&\leq f_\beta^p(r_0) \frac{1}{r_0^{p\beta-5/2}}.
\end{aligned}$$

Similarly we have

$$\left(\int_{r_0}^{4r_0} |z_2(r, t_0)|^2 dr \right)^{1/2} \lesssim \frac{f_\beta^p(r_0)}{r_0^{p\beta-5/2}}.$$

In summary we obtain

$$\left(\int_{r_0}^{4r_0} |\partial_t w(r, t_0)|^2 + |\partial_r w(r, t_0)|^2 dr \right)^{1/2} \leq C_p \frac{f_\beta^p(r_0)}{r_0^{p\beta-5/2}}. \quad (37)$$

The constant C_p depends on p only.

Remark The estimate above holds as long as $\beta \geq 2/(p-1)$ and the inequality

$$|u(x, t)| \leq \frac{f(r)}{|x|^\beta}$$

holds for all $|x| \geq r > 0$.

7.2 Recurrence Formula

We know $w = ru$ is a solution to the one-dimensional wave equation

$$\partial_t^2 w - \partial_r^2 w = r|u|^{p-1}u.$$

Using the explicit formula to solve this equation, we obtain

$$\begin{aligned}
r_0 u(r_0, t_0) &= \frac{1}{2} \left[\left(r_0 + \frac{r_0}{2} \right) u \left(r_0 + \frac{r_0}{2}, t_0 - \frac{r_0}{2} \right) + \left(r_0 - \frac{r_0}{2} \right) u \left(r_0 - \frac{r_0}{2}, t_0 - \frac{r_0}{2} \right) \right] \\
&\quad + \frac{1}{2} \int_{r_0 - \frac{r_0}{2}}^{r_0 + \frac{r_0}{2}} \partial_t w \left(r, t_0 - \frac{r_0}{2} \right) dr \\
&\quad + \frac{1}{2} \int_0^{\frac{r_0}{2}} \int_{\frac{r_0}{2} + t}^{\frac{3r_0}{2} - t} r |u|^{p-1} u \left(r, t_0 - \frac{r_0}{2} + t \right) dr dt \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

By Cauchy-Schwartz and (37), we have

$$\begin{aligned}
|I_2| &\leq \frac{1}{2} \left(\int_{\frac{r_0}{2}}^{\frac{3r_0}{2}} \left| \partial_t w \left(r, t_0 - \frac{r_0}{2} \right) \right|^2 dr \right)^{1/2} \left(\int_{\frac{r_0}{2}}^{\frac{3r_0}{2}} 1 dr \right)^{1/2} \\
&\leq C_p \frac{f_\beta^p(\frac{r_0}{2})}{r_0^{p\beta-5/2}} r_0^{1/2} \\
&= C_p f_\beta^p(\frac{r_0}{2}) r_0^{3-p\beta}.
\end{aligned}$$

Next we estimate I_3 using (36)

$$\begin{aligned}
|I_3| &\leq \frac{1}{2} \int_0^{r_0/2} \int_{r_0/2+t}^{3r_0/2-t} r \left(\frac{f_\beta(r_0/2)}{r^\beta} \right)^p dr dt \\
&\leq C_p \int_0^{r_0/2} r_0^2 \frac{f_\beta^p(r_0/2)}{r_0^{p\beta}} dt \\
&\leq C_p f_\beta^p(\frac{r_0}{2}) r_0^{3-p\beta}.
\end{aligned}$$

While

$$\begin{aligned}
|I_1| &\leq \frac{1}{2} \left[\frac{3r_0}{2} \frac{f_\beta(3r_0/2)}{(3r_0/2)^\beta} + \frac{r_0}{2} \frac{f_\beta(r_0/2)}{(r_0/2)^\beta} \right] \\
&= \frac{1}{2} \left[\left(\frac{3}{2} \right)^{1-\beta} f_\beta\left(\frac{3r_0}{2}\right) + \left(\frac{1}{2} \right)^{1-\beta} f_\beta\left(\frac{r_0}{2}\right) \right] r_0^{1-\beta}.
\end{aligned}$$

Combining these three terms and dividing both sides of the inequality by $r_0^{1-\beta}$, we obtain (replace r_0 by r)

$$r^\beta |u(r, t_0)| \leq \frac{1}{2} \left[\left(\frac{3}{2} \right)^{1-\beta} f_\beta\left(\frac{3r}{2}\right) + \left(\frac{1}{2} \right)^{1-\beta} f_\beta\left(\frac{r}{2}\right) \right] + C_p f_\beta^p\left(\frac{r}{2}\right) r^{2-(p-1)\beta}.$$

Observing that the right hand is a nonincreasing function of r , we apply $\sup_{r \geq r_0}$ on both sides and obtain

$$f_\beta(r_0) \leq \frac{1}{2} \left[\left(\frac{3}{2} \right)^{1-\beta} f_\beta\left(\frac{3r_0}{2}\right) + \left(\frac{1}{2} \right)^{1-\beta} f_\beta\left(\frac{r_0}{2}\right) \right] + C_p f_\beta^p\left(\frac{r_0}{2}\right) r_0^{2-(p-1)\beta}. \quad (38)$$

Thus

$$f_\beta(r_0) \leq \frac{1}{2} \left[\left(\frac{3}{2} \right)^{1-\beta} + \left(\frac{1}{2} \right)^{1-\beta} \right] f_\beta\left(\frac{r_0}{2}\right) + C_p f_\beta^p\left(\frac{r_0}{2}\right) r_0^{2-(p-1)\beta}. \quad (39)$$

7.3 Decay of $u(x, t)$

Let

$$g(\beta) = \frac{1}{2} \left[\left(\frac{3}{2} \right)^{1-\beta} + \left(\frac{1}{2} \right)^{1-\beta} \right] < 1.$$

Because $f_\beta(r) \rightarrow 0$ and $2 - (p-1)\beta \leq 0$, we know that there exists a large constant R , such that if $r_0 > R$,

$$C_p f_\beta^p\left(\frac{r_0}{2}\right) r_0^{2-(p-1)\beta} \leq \frac{1-g(\beta)}{2} f_\beta\left(\frac{r_0}{2}\right).$$

Thus we have for $r_0 > R$,

$$f_\beta(r_0) \leq \frac{g(\beta)+1}{2} f_\beta(r_0/2).$$

This implies

$$f_\beta(r) \leq C r^{\log_2\left(\frac{g(\beta)+1}{2}\right)}$$

for sufficiently large $r > R'$. As a result, for each $\beta_1 < \beta - \log_2\left(\frac{g(\beta)+1}{2}\right)$, we have (Note that the logarithm is negative)

$$|x|^{\beta_1} |u(x, t)| \leq f_\beta(|x|) |x|^{\beta_1-\beta} \leq C |x|^{\beta_1-\beta+\log_2\left(\frac{g(\beta)+1}{2}\right)} \rightarrow 0$$

as $|x| \rightarrow \infty$. This implies

$$\left[\beta, \beta + \log_2 \frac{2}{1+g(\beta)} \right) \subseteq U.$$

The Upper Bound of U Now we are ready to show $\sup U = 1$, if this was false, we could assume $\sup U = \beta_0 < 1$. Then we have for each $\beta \in U$,

$$g(\beta) \leq G_0 \doteq \max \left\{ g(\beta_0), g\left(\frac{2}{p-1}\right) \right\} < 1$$

using the convexity of the function g . Thus

$$\log_2 \frac{2}{1+g(\beta)} \geq \log_2 \frac{2}{1+G_0} > 0.$$

This means

$$\left[\beta, \beta + \log_2 \frac{2}{1+G_0} \right) \subseteq U.$$

This gives us a contradiction as $\beta \rightarrow \sup U$.

Decay of u Let β be a number slightly smaller than 1. We know $\beta \in U$. By (37), we have

$$\begin{aligned} \int_{r_0}^{4r_0} |\partial_r w(r, t_0)| dr &\leq \left(\int_{r_0}^{4r_0} |\partial_r w(r, t_0)|^2 dr \right)^{1/2} \left(\int_{r_0}^{4r_0} 1 dr \right)^{1/2} \\ &\leq \frac{C_p f_\beta^p(r_0)}{r_0^{p\beta-5/2}} r_0^{1/2} \\ &\leq \frac{C_{p,\beta}}{r_0^{p\beta-3}} \end{aligned}$$

We can choose $\beta \in U$ so that $p\beta - 3 > 0$ by the fact $p > 3$. Thus we have

$$\int_1^\infty |\partial_r w(r, t_0)| dr \leq C_{p,\beta}. \quad (40)$$

In addition for $r \leq 1$,

$$|w(r, t_0)| = r|u(r, t_0)| \leq C\|u(t_0)\|_{\dot{H}^{s_p}} r^{1-\frac{2}{p-1}} \leq C\|u(t_0)\|_{\dot{H}^{s_p}}.$$

Combining the two estimates above, we know that $|w(r, t)|$ is bounded by a universal constant C_1 for each pair (r, t) . Thus

$$|u(x, t)| \leq \frac{C_1}{|x|}. \quad (41)$$

Plugging this in the definition of $f_\beta(r)$, we have

$$f_\beta(r_0) = \sup_{|x| \geq r_0} |x|^\beta |u(x, t)| \leq \sup_{|x| \geq r_0} C_1 |x|^{\beta-1} = C_1 r_0^{\beta-1}.$$

Plugging this in (37), we obtain

$$\left(\int_{r_0}^{4r_0} |\partial_t w(r, t_0)|^2 + |\partial_r w(r, t_0)|^2 dr \right)^{1/2} \lesssim \frac{1}{r_0^{p-5/2}}. \quad (42)$$

By lemma 4.2, the estimates (41) and (42) imply

$$\int_{r < |x| < 4r} (|\nabla u(x, t)|^2 + |\partial_t u(x, t)|^2) dx \lesssim r^{-1}. \quad (43)$$

8 Death of Soliton-like Solution

8.1 Solitons in the Focusing Case

In order to kill the soliton-like minimal blow-up solutions, we need to consider the solitons of the wave equation. It turns out that there does not exist any soliton for our equation. The elliptic equation

$$-\Delta W(x) = |W(x)|^{p-1} W(x) \quad (44)$$

does admit a lot of radial solutions. However, none of these solutions is in the space \dot{H}^{s_p} . Among these solutions we are particularly interested in the solutions satisfying the same kind of property at infinity as (41).

Proposition 8.1. *The elliptic equation (44) has a solution $W_0(x)$ so that*

- $W_0(x)$ is a radial and smooth solution in $\mathbb{R}^3 \setminus \{0\}$.
- The point 0 is a singularity of $W_0(x)$.
- The solution $W_0(x)$ is NOT in the space $\dot{H}^{s_p}(\mathbb{R}^3)$.
- Its behavior near infinity is given by $(|x| > R_0)$

$$\left| W_0(x) - \frac{1}{|x|} \right| \leq \frac{C}{|x|^{p-2}}; \quad |\nabla W_0(x)| \leq \frac{C}{|x|^2}. \quad (45)$$

Please see the last section for a complete discussion of this solution.

Idea to deal with the soliton-like solutions We will show there does not exist a soliton-like minimal blow-up solution in the focusing case. This conclusion is natural because there is actually no soliton. However to prove this result is not an easy task. We will use a method developed by T. Duyckaerts, C. E. Kenig and F. Merle as I mentioned at the beginning of this paper. In their paper [4] they use this method to prove the soliton resolution conjecture for radial solutions of the focusing, energy-critical wave equation. The idea is to show that our soliton-like solution has to be so close to the solitons $\pm W_0(x)$ or their rescaled versions that they must be the same. But the soliton we mentioned above is not in the right space. This is a contradiction. In order to achieve this goal, we have to be able to understand the behaviour of a minimal blow-up solution if it is close to our soliton $W_0(x)$.

8.2 Preliminary Results

First of all, we recall a lemma proved in [5].

Lemma 8.2. (Energy channel) *Let $(v_0, v_1) \in \dot{H}^1 \times L^2$ be a pair of radial initial data. Suppose $v(x, t)$ is the solution of the linear wave equation with the given initial data (v_0, v_1) . Let $w(r, t) = rv(r, t)$ as usual, then for any $R > 0$ either the inequality*

$$\int_{|x| > R+t} (|\nabla v(x, t)|^2 + |\partial_t v(x, t)|^2) dx \geq 2\pi \int_R^\infty (|\partial_r w(r, 0)|^2 + |\partial_t w(r, 0)|^2) dr$$

holds for all $t > 0$, or

$$\int_{|x| > R-t} (|\nabla v(x, t)|^2 + |\partial_t v(x, t)|^2) dx \geq 2\pi \int_R^\infty (|\partial_r w(r, 0)|^2 + |\partial_t w(r, 0)|^2) dr$$

holds for all $t < 0$.

Definition of $V_R(x, t)$ Let us define ($R > 0$)

$$V_R(x, t) = \begin{cases} W_0(R + |t|), & \text{if } |x| \leq R + |t|; \\ W_0(|x|), & \text{if } |x| > R + |t|. \end{cases} \quad (46)$$

Now let us consider the norms of V_R . By (45), we have

$$W_0(x) \leq \frac{C_R}{|x|}.$$

for each $|x| \geq R$. Thus if $\frac{3}{r} + \frac{1}{q} < 1$,

$$\begin{aligned} \|V_R\|_{L^q L^r(\mathbb{R} \times \mathbb{R}^3)} &= \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} |V_R(x, t)|^r dx \right)^{q/r} dt \right)^{1/q} \\ &\lesssim \left(\int_{\mathbb{R}} \left((R + |t|)^3 |W_0(R + |t|)|^r + \int_{|x| > R+|t|} |W_0(x)|^r dx \right)^{q/r} dt \right)^{1/q} \end{aligned}$$

$$\begin{aligned}
&\lesssim C_R \left(\int_{\mathbb{R}} \left((R+|t|)^{3-r} + \int_{|x|>R+|t|} |x|^{-r} dx \right)^{q/r} dt \right)^{1/q} \\
&\lesssim_r C_R \left(\int_{\mathbb{R}} ((R+|t|)^{3-r})^{q/r} dt \right)^{1/q} \\
&\lesssim_{r,q} C_R \left(R^{(3-r)q/r+1} \right)^{1/q} \\
&\lesssim_{r,q} C_R R^{\frac{3}{r} + \frac{1}{q} - 1}.
\end{aligned}$$

Thus the following norms are all finite for $R > 0$.

$$\|V_R\|_{Y_{sp}(\mathbb{R})} < \infty; \quad \|V_R\|_{L^{2p/(p-3)}L^{2p}(\mathbb{R} \times \mathbb{R}^3)} < \infty.$$

Furthermore, if R is sufficiently large $R > R'$, we could choose $C_R = 2$, thus

$$\|V_R\|_{Y_{sp}(\mathbb{R})} \lesssim R^{\frac{1}{2}-sp}; \quad \|V_R\|_{L^{2p/(p-3)}L^{2p}(\mathbb{R} \times \mathbb{R}^3)} \lesssim R^{-1/2}. \quad (47)$$

8.3 Approximation Theory

Theorem 8.3. *Fix $3 < p < 5$. There exists a constant $\delta_0 > 0$, such that if $\delta < \delta_0$ and we have*

- (i) *A function $V(x, t) \in L^{2p/(p-3)}L^{2p}(I \times \mathbb{R}^3)$ with $\|V(x, t)\|_{Y_{sp}(I)} < \delta$. Here I is a time interval containing 0;*
- (ii) *A pair of initial data (h_0, h_1) with*

$$\|(h_0, h_1)\|_{\dot{H}^1 \times L^2(\mathbb{R}^3)} < \delta, \quad \|(h_0, h_1)\|_{\dot{H}^{sp} \times \dot{H}^{sp-1}(\mathbb{R}^3)} < \delta.$$

Then the equation

$$\begin{cases} \partial_t^2 h - \Delta h = F(V + h) - F(V), & (x, t) \in \mathbb{R}^3 \times I; \\ h|_{t=0} = h_0; \\ \partial_t h|_{t=0} = h_1 \end{cases}$$

has a unique solution $h(x, t)$ on $I \times \mathbb{R}^3$ so that

$$\|h\|_{Y_{sp}(I)} \leq C_p \delta;$$

$$\sup_{t \in I} \|(h, \partial_t h) - (h_L, \partial_t h_L)\|_{\dot{H}^1 \times L^2} \leq C_p \delta^{p-1} \|(h_0, h_1)\|_{\dot{H}^1 \times L^2}.$$

Here $(h_L, \partial_t h_L)$ is the solution of the linear wave equation with initial data (h_0, h_1) .

Stretch of Proof In this proof C_p represents a constant that depends on p only. In different places C_p may represent different constants. We will also write Y instead of $Y_{sp}(I)$ for convenience. By the Strichartz estimates, we have

$$\|F(V + h) - F(V)\|_{Z_{sp}} \leq C_p \|h\|_Y (\|h\|_Y^{p-1} + \|V\|_Y^{p-1});$$

$$\|F(V + h^{(1)}) - F(V + h^{(2)})\|_{Z_{sp}} \leq C_p \|h^{(1)} - h^{(2)}\|_Y (\|h^{(1)}\|_Y^{p-1} + \|h^{(2)}\|_Y^{p-1} + \|V\|_Y^{p-1}).$$

By a fixed point argument, if δ is sufficiently small, we have a unique solution $h(x, t)$ defined on $I \times \mathbb{R}^3$, so that $\|h\|_Y \leq C_p \delta$. Now by Strichartz estimates, if δ is sufficiently small

$$\begin{aligned}
\|h\|_{L^{\frac{4p}{9-p}} L^{\frac{4p}{p-3}}} &\leq C_p (\|(h_0, h_1)\|_{\dot{H}^1 \times L^2} + \|F(V+h) - F(V)\|_{L^1 L^2}) \\
&\leq C_p \left(\|(h_0, h_1)\|_{\dot{H}^1 \times L^2} + \|h\|_{L^{\frac{4p}{9-p}} L^{\frac{4p}{p-3}}} (\|h\|_Y^{p-1} + \|V\|_Y^{p-1}) \right) \\
&\leq C_p \|(h_0, h_1)\|_{\dot{H}^1 \times L^2} + C_p \delta^{p-1} \|h\|_{L^{\frac{4p}{9-p}} L^{\frac{4p}{p-3}}} \\
&\leq C_p \|(h_0, h_1)\|_{\dot{H}^1 \times L^2} + (1/2) \|h\|_{L^{\frac{4p}{9-p}} L^{\frac{4p}{p-3}}}
\end{aligned}$$

Thus

$$\|h\|_{L^{\frac{4p}{9-p}} L^{\frac{4p}{p-3}}} \leq C_p \|(h_0, h_1)\|_{\dot{H}^1 \times L^2}.$$

This gives us

$$\begin{aligned}
\sup_{t \in I} \|(h, \partial_t h) - (h_L, \partial_t h_L)\|_{\dot{H}^1 \times L^2} &\leq C_p \|F(V+h) - F(V)\|_{L^1 L^2} \\
&\leq C_p \|h\|_{L^{\frac{4p}{9-p}} L^{\frac{4p}{p-3}}} (\|h\|_Y^{p-1} + \|V\|_Y^{p-1}) \\
&\leq C_p \delta^{p-1} \|(h_0, h_1)\|_{\dot{H}^1 \times L^2}.
\end{aligned}$$

8.4 Match with $W_0(x)$

Using the estimate (42), we have

$$\int_{r_0}^{4r_0} |\partial_r w(r, t)| dr \lesssim \left(\int_{r_0}^{4r_0} |\partial_r w(r, t)|^2 dr \right)^{1/2} r_0^{1/2} \lesssim \frac{1}{r_0^{p-3}}.$$

This means

$$\int_{r_0}^{\infty} |\partial_r w(r, t)| dr \lesssim \frac{1}{r_0^{p-3}}. \quad (48)$$

Thus we know the limit $\lim_{r \rightarrow \infty} w(r, t)$ exists for each t .

Case 1 If $\lim_{r \rightarrow \infty} w(r, 0) = 0$. Then in the rest of this section, set $W(x) = 0$. By (48) we have

$$|w(r, 0)| \lesssim \frac{1}{r^{p-3}}.$$

Thus

$$|u_0(x) - W(x)| = \frac{1}{|x|} |w(|x|, 0)| \lesssim \frac{1}{|x|^{p-2}}.$$

Case 2 If $\lim_{r \rightarrow \infty} w(r, 0) \neq 0$. WLOG, let us assume the limit is equal to 1. Otherwise we only need to apply some space-time dilation and/or multiplication by -1 on u .

In the rest of this section, set $W(x) = W_0(x)$. Thus by (48) we have

$$|w(r, 0) - 1| \leq \int_r^{\infty} |\partial_r w(r, t)| dr \lesssim \frac{1}{r^{p-3}}.$$

Dividing this inequality by r , we have

$$\left| u_0(x) - \frac{1}{|x|} \right| \lesssim \frac{1}{|x|^{p-2}}.$$

Combining this with our estimate for $W_0(x)$, we have for large x

$$|u_0(x) - W(x)| \lesssim \frac{1}{|x|^{p-2}}.$$

8.5 Identity near infinity

Theorem 8.4. *Let $W(x) = W_0(x)$ or $W(x) = 0$. Suppose $u(x, t)$ is a global radial solution of the equation (1) with initial data (u_0, u_1) satisfying the following conditions.*

(I) $(u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}$.

(II) *The following inequality holds for each $t \in \mathbb{R}$ and $r > 0$.*

$$\int_{r < |x| < 4r} (|\nabla u(x, t)|^2 + |\partial_t u(x, t)|^2) dx \leq C_1 r^{-1}. \quad (49)$$

(III) *We have $u_0(x)$ and $W(x)$ are very close to each other as $|x|$ is large.*

$$|u_0(x) - W(x)| \lesssim \frac{1}{|x|^{p-2}}. \quad (50)$$

Then there exists $R_0 = R_0(C_1, p)$ such that the pair $(u_0(x) - W(x), u_1(x))$ is essentially supported in the ball $B(0, R_0)$.

Remark There are actually two separate theorems, both could be proved in the same way. If $W(x) = W_0(x)$ (the primary case), then define V_{R_0} as usual in the proof below. Otherwise if $W(x) = 0$, just make $V_{R_0} = 0$.

Proof Let us define for $R \geq R_0$

$$g_0 = \Psi_R(u_0 - W); \quad g_1 = \Psi_R u_1.$$

$$G(r) = u_0(r) - W(r).$$

Choose a small constant $\delta = \delta(p)$, so that it is smaller than the constant δ_0 in theorem 8.3 and guarantees the number $C_p \delta^{p-1}$ in the conclusion of that theorem is smaller than $\varepsilon(p)$, which is a small number determined later in the argument below. By the condition (49) and the properties of $W(x)$, we know ($R > 1$)

$$\begin{aligned} \int_{\mathbb{R}^3} (|\nabla g_0|^2 + g_1^2) dx &\lesssim_{C_1, p} R^{-1}; \\ \int_{\mathbb{R}^3} \left(|\nabla g_0|^{3(p-1)/(p+1)} + g_1^{3(p-1)/(p+1)} \right) dx &\lesssim_{C_1, p} R^{-3(p-3)/(p+1)}. \end{aligned}$$

As a result, if $R_0 = R_0(C_1, p)$ is sufficiently large, we have the following inequalities hold as long as $R \geq R_0$. (We use the Sobolev embedding in order to obtain the second inequality)

$$\|(g_0, g_1)\|_{\dot{H}^1 \times L^2} \leq \delta; \quad \|(g_0, g_1)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} \leq \delta;$$

$$\|V_{R_0}\|_{Y_{sp}(\mathbb{R})} \leq \delta.$$

Let g be the solution of

$$\partial_t^2 g - \Delta g = F(V_{R_0} + g) - F(V_{R_0})$$

with the initial data (g_0, g_1) and \tilde{g} be the solution of the linear wave equation with the same initial data. On the other hand, we know $u(x, t) - W(x)$ is the solution of the equation

$$\partial_t^2 \tilde{u} - \Delta \tilde{u} = F(W + \tilde{u}) - F(W) \quad (51)$$

in the domain $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$ with the initial data $(u_0 - W, u_1)$. Let K be the domain

$$K = \{(x, t) : |x| > |t| + R\}.$$

Considering the fact $W(x) = V_{R_0}(x, t)$ in the region K and the construction of (g_0, g_1) , we have

$$u(x, t) - W(x) = g(x, t); \quad \partial_t u(x, t) = \partial_t g(x, t)$$

in the domain K by the finite speed of propagation. Using our assumption (49) and the decay of $W(x)$ at infinity, we have

$$\lim_{t \rightarrow \pm\infty} \int_{|x| > |t| + R} (|\nabla g(x, t)|^2 + |\partial_t g(x, t)|^2) dx \rightarrow 0. \quad (52)$$

Using lemma 8.2, WLOG, let us assume for all $t > 0$

$$\int_{|x| > R+t} (|\nabla \tilde{g}(x, t)|^2 + |\partial_t \tilde{g}(x, t)|^2) dx \geq 2\pi \int_R^\infty (|\partial_r(r g_0(r, 0))|^2 + r^2 |g_1(r, 0)|^2) dr.$$

That is

$$\int_{|x| > R+t} (|\nabla \tilde{g}(x, t)|^2 + |\partial_t \tilde{g}(x, t)|^2) dx \geq \frac{1}{2} \left(\int_{|x| > R} (|\nabla g_0|^2 + g_1^2) dx \right) - 2\pi R g_0^2(R).$$

Combining this with (52), we have

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \|(g(x, t), \partial_t g(x, t)) - (\tilde{g}, \partial_t \tilde{g})\|_{\dot{H}^1 \times L^2(|x| > R+t)} \\ & \geq \left(\frac{1}{2} \int_{|x| > R} (|\nabla g_0|^2 + g_1^2) dx - 2\pi R g_0^2(R) \right)^{1/2}. \end{aligned}$$

On the other hand, we have the following inequality by theorem 8.3

$$\|(g(x, t), \partial_t g(x, t)) - (\tilde{g}, \partial_t \tilde{g})\|_{\dot{H}^1 \times L^2} \leq C_p \delta^{p-1} \|(g_0, g_1)\|_{\dot{H}^1 \times L^2} \leq \varepsilon(p) \|(g_0, g_1)\|_{\dot{H}^1 \times L^2}.$$

Considering both inequalities above, we have

$$\frac{1}{2} \int_{|x| > R} (|\nabla g_0|^2 + g_1^2) dx - 2\pi R g_0^2(R) \leq \varepsilon^2(p) \int_{|x| > R} (|\nabla g_0|^2 + g_1^2) dx.$$

Thus

$$\int_{|x| > R} (|\nabla g_0|^2 + g_1^2) dx \leq \frac{4\pi}{1 - 2\varepsilon^2(p)} R g_0^2(R). \quad (53)$$

We have

$$\begin{aligned}
|g_0(mR) - g_0(R)| &\leq \int_R^{mR} |\partial_r g_0| dr \\
&\leq \left(\int_R^{mR} |r \partial_r g_0|^2 dr \right)^{1/2} \left(\int_R^{mR} \frac{1}{r^2} dr \right)^{1/2} \\
&\leq \left(\frac{1}{4\pi} \int_{|x|>R} (|\nabla g_0|^2 + g_1^2) dx \right)^{1/2} \left(\frac{1}{R} - \frac{1}{mR} \right)^{1/2} \\
&\leq \left(\frac{R g_0^2(R)}{1 - 2\varepsilon^2(p)} \right)^{1/2} \left(1 - \frac{1}{m} \right)^{1/2} R^{-1/2} \\
&\leq \left(\frac{1 - 1/m}{1 - 2\varepsilon^2(p)} \right)^{1/2} |g_0(R)|.
\end{aligned}$$

By the fact $p - 2 > 1$, we can choose $k = k(p) \in \mathbb{Z}^+$ such that $(k + 1)/k < p - 2$. Let $m = 2^k$. Since

$$(1 - 1/m)^{1/2} < 1 - \frac{1}{2m},$$

we can choose $\varepsilon(p)$ so small that

$$\left(\frac{1 - 1/m}{1 - 2\varepsilon^2(p)} \right)^{1/2} \leq 1 - \frac{1}{2m} = 1 - \frac{1}{2^{k+1}}.$$

Plugging this in our estimate above, we obtain

$$|g_0(2^k R) - g_0(R)| \leq (1 - \frac{1}{2^{k+1}}) |g_0(R)|.$$

Thus

$$|g_0(2^k R)| \geq \frac{1}{2^{k+1}} |g_0(R)|.$$

By the definition of g_0 , this is the same as

$$|G(2^k R)| \geq \frac{1}{2^{k+1}} |G(R)|.$$

This inequality holds for all $R \geq R_0$. Now let us consider the value of $G(R_0)$. If $G(R_0) = 0$, let us choose $R = R_0$. Thus we have $g_0(R) = 0$. Plugging this back in (53), we have $(g_0, g_1) = (0, 0)$. This means that $(u_0 - W, u_1)$ is supported in $B(0, R_0)$ and finishes the proof. If $|G(R_0)| > 0$, then we have

$$|G(2^{kn} R_0)| \geq \frac{1}{(2^{kn})^{(k+1)/k}} |G(R_0)| > 0$$

for each positive integer n . This contradicts with the condition (50) because $(k+1)/k < p-2$ by our choice of k .

Remark If one feels uncomfortable about the singularity at zero in the equation (51), we could use the following center-cutoff version instead. Let φ be a smooth, radial, nonnegative function satisfying

$$\varphi(x) = \begin{cases} 1, & \text{if } |x| \geq 1; \\ \in [0, 1], & \text{if } |x| \in (1/2, 1); \\ 0, & \text{if } |x| \leq 1/2. \end{cases}$$

Then $u(x, t) - \varphi(|x|/R_0)W_0(x)$ is a solution to the equation

$$\begin{cases} \partial_t^2 \tilde{u} - \Delta \tilde{u} = F(\varphi(|x|/R_0)W_0 + \tilde{u}) + \Delta(\varphi(|x|/R_0)W_0(x)), & (x, t) \in \mathbb{R}^3 \times \mathbb{R}; \\ \tilde{u}|_{t=0} = u_0 - \varphi(|x|/R_0)W_0 \in \dot{H}^{s_p}(\mathbb{R}^3); \\ \partial_t \tilde{u}|_{t=0} = u_1 \in \dot{H}^{s_p-1}(\mathbb{R}^3). \end{cases}$$

For any $T > 0$, we know

$$\|\varphi(|x|/R_0)W_0(x)\|_{Y_{s_p}([-T, T])} < \infty; \quad \|\Delta(\varphi(|x|/R_0)W_0(x))\|_{Z_{s_p}([-T, T])} < \infty.$$

In addition, the function $\Delta(\varphi(|x|/R_0)W_0(x)) = -F(W_0(x))$ in the region K . We can do the argument as usual in the proof above but avoid the singularity at zero with this new cutoff version of the equation (51). This method also works in the proof of theorem 8.5, which will be introduced in the next subsection.

Application of the theorem Now apply theorem 8.4 to our soliton-like minimal blow-up solution. All the conditions are satisfied by our earlier argument. Thus $(u_0(x) - W(x), u_1(x))$ is supported in the ball of radius R_0 centered at the origin. In particular, because R_0 depends only on the constant C_1 and p , the same R_0 also works for other time t as long as the condition (50) is true at that time. But by the finite speed of propagation, we know $(u(x, t) - W(x), \partial_t u(x, t))$ is compactly supported in $B(0, R_0 + |t|)$ at each time t . This means the condition (50) is always true at any time. Thus the pair $(u(x, t) - W(x), \partial_t u)$ is supported in the cylinder $B(0, R_0) \times \mathbb{R}$.

8.6 Local Radius Analysis

Let us define the essential radius of the support of $(u(x, t) - W(x), \partial_t u(x, t))$ at time t .

$$R(t) = \min\{R \geq 0 : (u(x, t) - W(x), \partial_t u(x, t)) = (0, 0) \text{ holds for } |x| > R\}.$$

This is well-defined for our minimal blow-up solution. Actually $R(t) \leq R_0$ holds for all t .

Theorem 8.5. (Behavior of compactly supported solutions) *Let $W(x) = W_0(x)$ or $W(x) = 0$. Let $u(x, t)$ be a radial solution of the equation (1) in a time interval I , so that*

- (I) The pair $(u(x, t), \partial_t u(x, t)) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ for each $t \in I$.*
- (II) The pair $(u(x, 0) - W(x), \partial_t u(x, 0))$ is compactly supported with an essential radius of support $R(0) > R_1 > 0$. Then there exists a constant $\tau = \tau(R_1, p)$, such that*

$$R(t) = R(0) + |t|$$

holds either for each $t \in [0, \tau] \cap I$ or for each $t \in [-\tau, 0] \cap I$.

Remark If $W(x) = W_0(x)$ (the primary case), then define V_{R_1} as usual in the proof. Otherwise if $W(x) = 0$, just make $V_{R_1} = 0$. In this case we can choose $\tau = \infty$.

Proof By our previous argument, we have $\|V_{R_1}\|_{Y_{sp}(\mathbb{R})} < \infty$. Thus we can choose $\tau = \tau(R_1, p) > 0$ such that $\|V_{R_1}\|_{Y_{sp}([- \tau, \tau])} < \delta$. Here δ is a small constant so that we can apply theorem 8.3 and make the number $C_p \delta^{p-1} < 1/100$ in that theorem. If $\varepsilon < R(0) - R_1$, let us define a pair of initial data (g_0, g_1) for each $R \in (R(0) - \varepsilon, R(0))$

$$g_0 = \Psi_R(u_0 - W); \quad g_1 = \Psi_R u_1.$$

This pair $(g_0(x), g_1(x))$ is nonzero by the definition of $R(0)$.

By our assumptions on (u_0, u_1) , we know the following inequalities hold for each $R \in (R(0) - \varepsilon, R(0))$ as long as ε is sufficiently small. (In order to obtain the second inequality we use the Sobolev embedding)

$$\|(g_0, g_1)\|_{\dot{H}^1 \times L^2} < \delta;$$

$$\|(g_0, g_1)\|_{\dot{H}^{sp} \times \dot{H}^{sp-1}} < \delta.$$

Furthermore, we have

$$\begin{aligned} |g_0(R)| &= \left| g_0(R(0)) - \int_R^{R(0)} \partial_r g_0(r) dr \right| \\ &\leq \int_R^{R(0)} |\partial_r g_0(r)| dr \\ &\leq \left(\int_R^{R(0)} r^2 |\partial_r g_0(r)|^2 dr \right)^{1/2} \left(\int_R^{R(0)} \frac{1}{r^2} dr \right)^{1/2} \\ &\leq \left(\int_R^{R(0)} r^2 |\partial_r g_0(r)|^2 dr \right)^{1/2} \left(\frac{R(0) - R}{R(0)R} \right)^{1/2} \\ &\leq \left(\frac{\varepsilon}{R(0)R} \int_R^{R(0)} r^2 |\partial_r g_0(r)|^2 dr \right)^{1/2}. \end{aligned}$$

Thus

$$Rg_0^2(R) \leq \frac{\varepsilon}{R(0)} \int_R^{R(0)} r^2 |\partial_r g_0(r)|^2 dr \leq \frac{\varepsilon}{4\pi R(0)} \int_{R < |x| < R(0)} (|\nabla g_0(x)|^2 + |g_1(x)|^2) dx.$$

If ε is sufficiently small, we can apply lemma 4.2 to obtain

$$\int_R^{R(0)} \left[|\partial_r(r g_0(r))|^2 + r^2 |g_1(r)|^2 \right] dr \geq \frac{0.99}{4\pi} \int_{R < |x| < R(0)} (|\nabla g_0(x)|^2 + |g_1(x)|^2) dx.$$

Let $\tilde{g}(x, t)$ be the solution to the linear wave equation with the initial data (g_0, g_1) . By lemma 8.2,

$$\int_{|x| > R+|t|} (|\nabla \tilde{g}(x, t)|^2 + |\partial_t \tilde{g}(x, t)|^2) dx \geq 2\pi \int_R^\infty \left[|\partial_r(r g_0(r))|^2 + r^2 |g_1(r)|^2 \right] dr$$

$$\begin{aligned}
&= 2\pi \int_R^{R(0)} \left[|\partial_r(r g_0(r))|^2 + r^2 |g_1(r)|^2 \right] dr \\
&\geq 0.49 \int_{R < |x| < R(0)} (|\nabla g_0(x)|^2 + |g_1(x)|^2) dx
\end{aligned}$$

holds either for each $t \geq 0$ or for each $t \leq 0$. WLOG, let us choose $t \geq 0$, then we have

$$\|(\tilde{g}(x, t), \partial_t \tilde{g}(x, t))\|_{\dot{H}^1 \times L^2(|x| > R+t)} \geq 0.7 \|(g_0, g_1)\|_{\dot{H}^1 \times L^2(\mathbb{R}^3)}. \quad (54)$$

Let g be the solution of the following equation

$$\begin{cases} \partial_t^2 g - \Delta g = F(V_{R_1} + g) - F(V_{R_1}), & (x, t) \in \mathbb{R}^3 \times I; \\ g|_{t=0} = g_0; \\ \partial_t g|_{t=0} = g_1. \end{cases}$$

By lemma 8.3, we have

$$\|(g(x, t), \partial_t g(x, t)) - (\tilde{g}(x, t), \partial_t \tilde{g}(x, t))\|_{\dot{H}^1 \times L^2} \leq 0.01 \|(g_0, g_1)\|_{\dot{H}^1 \times L^2(\mathbb{R}^3)}$$

for each $t \in [-\tau, \tau]$. Combining this with (54), for $t \in [0, \tau]$ we obtain

$$\|(g(x, t), \partial_t g(x, t))\|_{\dot{H}^1 \times L^2(|x| > R+t)} \geq 0.69 \|(g_0, g_1)\|_{\dot{H}^1 \times L^2(\mathbb{R}^3)}. \quad (55)$$

In addition, we know $u(x, t) - W(x)$ is the solution of equation

$$\begin{cases} \partial_t^2 \tilde{u} - \Delta \tilde{u} = F(W(x) + \tilde{u}) - F(W(x)), & (x, t) \in \mathbb{R}^3 \times I; \\ \tilde{u}|_{t=0} = u_0 - W; \\ \partial_t \tilde{u}|_{t=0} = u_1 \end{cases}$$

in $(\mathbb{R}^3 \setminus \{0\}) \times I$. The initial data of these two equations is the same in the region $\{x : |x| \geq R\}$ and the nonlinear part is the same function in the region

$$K = \{(x, t) : |x| > R + t, t \in [0, \tau] \cap I\}$$

Thus by the finite speed of propagation, we have $g(x, t) = u(x, t) - W(x)$ and $\partial_t g(x, t) = \partial_t u(x, t)$ in K . Plugging this in (55), we obtain

$$\|(u(x, t) - W(x), \partial_t u(x, t))\|_{\dot{H}^1 \times L^2(|x| > R+t)} \geq 0.69 \|(g_0, g_1)\|_{\dot{H}^1 \times L^2(\mathbb{R}^3)}$$

for each $t \in I \cap [0, \tau]$. Since $R < R(0)$, we know the right hand of the inequality above is positive by the definition of essential radius of support. Thus we have

$$R(t) \geq R + |t| \quad (56)$$

for all $t \in [0, \tau] \cap I$. Letting $R \rightarrow R(0)^-$, we obtain $R(t) \geq R(0) + |t|$. By the finite speed of propagation, we have $R(t) = R(0) + |t|$.

Remark For each $R \in (R(0) - \varepsilon, R(0))$, we know that the inequality (56) above holds either in the positive or negative time direction. It may work in different directions as we choose different R 's. However, we can always choose a sequence $R_i \rightarrow R(0)^-$ such that the inequality works in the same time direction for all R_i 's. This is sufficient for us to conclude the theorem.

8.7 End of Soliton-like Solution

Now let us show $R(0) = 0$. If it was not zero, let $R_1 = R(0)/2$, and then apply theorem 8.5. We have (WLOG) $R(t) = R(0) + t$ for each $t \in [0, \tau]$. Applying theorem 8.5 again at $t = \tau$, we obtain

$$R(t) = R(0) + \tau + (t - \tau) = R(0) + t$$

for $t \in [\tau, 2\tau]$, because (i) The same constant τ works by $R(\tau) > R(0) > R_1$; (ii) The theorem may only work in the positive time direction, since we know the radius of support $R(t)$ decreases in the other direction. Repeating this process, we have for each $t > 0$,

$$R(t) = R(0) + t.$$

But it is impossible since $R(t)$ is uniformly bounded by R_0 . Therefore we must have $R(0) = 0$. But this means either $u_0 = W_0(x) \notin \dot{H}^{s_p}(\mathbb{R}^3)$ or $(u_0, u_1) = (0, 0)$. This is a contradiction.

9 The Solution of the Elliptic Equation

In this section we will consider the elliptic equation

$$-\Delta W(x) = |W(x)|^{p-1}W(x). \quad (57)$$

It has infinitely many solutions. For example,

$$W_1(x) = C|x|^{-2/(p-1)}$$

is a solution if we choose an appropriate constant C . We are interested in radial solutions of this elliptic equation. Let us assume $W(x) = y(|x|)$. Here $y(r)$ is a function defined in $(0, \infty)$. The function $y(r)$ satisfies the equation

$$y''(r) + \frac{2}{r}y'(r) + |y|^{p-1}y(r) = 0. \quad (58)$$

Let us show that the solution $W_0(x)$ we mentioned earlier in this paper exists.

9.1 Existence of $W_0(x)$

We are seeking a solution with the property $W_0(x) \simeq 1/|x|$ as x is large. That is equivalent to $y(r) \simeq 1/r$. Let us define $\rho(r) = ry(r)$, then $\rho(r)$ satisfies

$$\rho''(r) = -\frac{F(\rho)}{r^{p-1}}; \quad F(\rho) = |\rho|^{p-1}\rho.$$

We expect $\rho(r) \simeq 1$ for large r 's, thus let us assume $\rho(r) = \phi(r) + 1$. The corresponding equation for $\phi(r)$ is given as below

$$\phi''(r) = -\frac{F(\phi + 1)}{r^{p-1}}.$$

The idea is to show

- (I) This equation has a solution in the interval $[R, \infty)$ with boundary conditions at infinity $\phi(+\infty) = \phi'(+\infty) = 0$, by a fixed point argument.
- (II) We can expand the domain of this solution to \mathbb{R}^+ .

The Fixed Point Argument Let us consider the metric space

$$K = \{\phi : \phi \in C([R, \infty); [-1, 1]), \lim_{r \rightarrow +\infty} \phi(r) = 0\}.$$

with the distance $d(\phi_1, \phi_2) = \sup_r |\phi_1(r) - \phi_2(r)|$. One can check K is complete. Let us define a map $L : K \rightarrow K$ by

$$L(\phi)(r) = \int_r^\infty \left(\int_s^\infty \left(-\frac{F(\phi(t) + 1)}{t^{p-1}} \right) dt \right) ds.$$

We have

$$\begin{aligned} |L(\phi)(r)| &\leq \int_r^\infty \left(\int_s^\infty \left(\frac{2^p}{t^{p-1}} \right) dt \right) ds \leq \frac{C_p}{r^{p-3}}; \\ |L(\phi_1)(r) - L(\phi_2)(r)| &\leq C_p \int_r^\infty \left(\int_s^\infty \left(\frac{d(\phi_1, \phi_2)}{t^{p-1}} \right) dt \right) ds \leq C_p \frac{d(\phi_1, \phi_2)}{r^{p-3}}. \end{aligned}$$

Thus if $R > R(p)$ is a sufficiently large number, then L is a contraction map from K to itself. As a result, there exists a unique fixed point $\phi_0(r)$. This gives us a classic smooth solution of the ODE in $[R, \infty)$. We have $\phi_0(r) \lesssim r^{3-p}$ and its derivative $\phi'_0(r)$ satisfies

$$|\phi'_0(r)| = \left| \int_r^\infty \left(\frac{F(\phi_0(t) + 1)}{t^{p-1}} \right) dt \right| \leq \frac{C_p}{r^{p-2}}.$$

Expansion of the Solution Now let us solve the ODE backward from $r = R$. We need to show it will never break down before we approach zero. Actually we have

$$\frac{d}{dr} \left(\frac{|\phi_0 + 1|^{p+1}}{p+1} + \frac{r^{p-1}|\phi'_0|^2}{2} \right) = \frac{p-1}{2} r^{p-2} |\phi'_0|^2 \geq 0.$$

Thus we have the following inequality holds for all $0 < r \leq R$ as long as the solution still exists at r

$$\frac{|\phi_0(r) + 1|^{p+1}}{p+1} + \frac{r^{p-1}|\phi'_0(r)|^2}{2} \leq C.$$

But this implies the solution will never break down at a positive r . Let us define

$$W_0(x) = \frac{\phi_0(|x|) + 1}{|x|}.$$

This is a C^2 , radial solution of our elliptic equation for $|x| > 0$. Furthermore, we have for large x

$$\begin{aligned} \left| W_0(x) - \frac{1}{|x|} \right| &= \frac{|\phi_0(|x|)|}{|x|} \leq \frac{C_p}{|x|^{p-2}}; \\ |\nabla W_0(x)| &= \left| \frac{r\phi'_0(r) - \phi_0(r) - 1}{r^2} \right|_{r=|x|} \leq \frac{C_p}{|x|^2}. \end{aligned}$$

Now the remaining task is to show $W_0(x)$ is not in the space \dot{H}^{s_p} . This implies $W_0(x)$ must have a singularity at 0. It turns out that it is not trivial. For instance, if we repeat the argument as above in the case $p = 5$, then the solution we obtain will be a smooth function in the whole space, as below

$$W(x) = \frac{\sqrt{3}}{(1 + 3|x|^2)^{1/2}}.$$

9.2 Radial \dot{H}^{s_p} Solution Does Not Exist

Regularity Let us first show any radial \dot{H}^{s_p} solution of the elliptic equation must be in the space $C^2(\mathbb{R}^3 \setminus \{0\})$. We know a radial \dot{H}^{s_p} function must be continuous except for $x = 0$. Using this in the elliptic equation, we have the solution is C^2 except for $x = 0$.

Introduction to $r^\theta y(r)$ We assume $W(x) = y(|x|)$. The function $y(r)$ defined in \mathbb{R}^+ is a C^2 solution of

$$y''(r) + \frac{2}{r}y'(r) + |y|^{p-1}y(r) = 0.$$

Let us define another $C^2(\mathbb{R}^+)$ function

$$v(r) = r^\theta y(r), \quad \theta = \frac{2}{p-1}.$$

If $W(x) = y(|x|)$ is in the space \dot{H}^{s_p} , then we have

$$\lim_{r \rightarrow 0^+} v(r) = \lim_{r \rightarrow +\infty} v(r) = 0.$$

Plugging $y(r) = r^{-\theta}v(r)$ in the equation for $y(r)$, we obtain an equation for $v(r)$,

$$r^2 v''(r) + \frac{2(p-3)}{p-1} r v'(r) - \frac{2(p-3)}{(p-1)^2} v(r) + |v|^{p-1} v(r) = 0.$$

Multiplying both sides by $v'(r)$, we obtain

$$\frac{d}{dr} \left(r^2 \frac{|v'(r)|^2}{2} - \frac{p-3}{(p-1)^2} v^2(r) + \frac{|v(r)|^{p+1}}{p+1} \right) = \frac{5-p}{p-1} r |v'(r)|^2 \geq 0. \quad (59)$$

These identities imply

Claim 1 There exist $r_1, R_1 > 0$, such that the function $v(r)$ does not admit any positive local maximum or negative local minimum in the set $K = (0, r_1) \cup (R_1, \infty)$. Actually by the limits of the function $v(r)$ at 0^+ and $+\infty$, we know there exist $r_1, R_1 > 0$, such that $v(r) \leq \varepsilon$ for $r \in K$. If ε is sufficiently small, then the sign of the sum (in the equation)

$$-\frac{2(p-3)}{(p-1)^2} v(r) + |v|^{p-1} v(r)$$

is the same as $-v(r)$. If there was a positive local maximum or a negative local minimum in K , we would find a contradiction by considering the sign of $v''(r)$, $v'(r)$ and $v(r)$.

Claim 2 Let r_1, R_1 be the constants in Claim 1. Then in each of the intervals $(0, r_1)$ and $(R_1, +\infty)$, the function $v(r)$ is monotone. (Namely it is nondecreasing or nonincreasing) Suppose this was not true. WLOG, let us assume $s_1 < s_2 < s_3$ and $v(s_2) < v(s_3), v(s_1)$. If we like, we can choose s_2 to be a local minimum (a minimum in the interval $[s_1, s_3]$). Then by Claim 1, $v(s_2)$ must be nonnegative. Now we obtain

- (i) If $s_i < r_1$, then there must be a positive local maximum in $(0, s_2)$;
- (ii) If $s_i > R_1$, then this yields a positive local maximum in (s_2, ∞) .

In both cases we have a contradiction.

Claim 3 If $v(r)$ is not the zero function, then at least one of the following inequality holds

$$\liminf_{r \rightarrow 0^+} r^2 |v'(r)|^2 > 0. \quad \liminf_{r \rightarrow +\infty} r^2 |v'(r)|^2 > 0.$$

If both of these failed, by considering the integral of (59) in the interval $r = (\varepsilon, M)$ and letting $\varepsilon \rightarrow 0^+$ and $M \rightarrow +\infty$, we would have

$$\frac{5-p}{p-1} \int_0^\infty r |v'(r)|^2 dr = 0$$

This means $v'(r) = 0$ everywhere, so $v(r) = 0$. But we assume it is not the zero function.

Contradiction If $v(r)$ is not identically zero, WLOG, let us assume

$$\liminf_{r \rightarrow 0^+} r^2 |v'(r)|^2 > 0.$$

This means there exist $C > 0$ and $r_2 > 0$, such that if $r \in (0, r_2)$, we have $r^2 |v'(r)|^2 > C$. This means $|v'(r)| > \sqrt{C} r^{-1}$. By Claim 2 $v'(r)$ does not change its sign in the interval $(0, r_1)$. Combining this fact with the lower bound of $|v'(r)|$, we know the limit of $v(r)$ does not exist at 0^+ . This gives us a contradiction. Therefore we have

Theorem 9.1. *If $3 < p < 5$, then a radial $\dot{H}^{sp}(\mathbb{R}^3)$ solution to the elliptic equation*

$$-\Delta W(x) = |W(x)|^{p-1} W(x)$$

must be the zero solution.

Conclusion In summary, any radial nontrivial solution of our elliptic equation is not in the space $\dot{H}^{sp}(\mathbb{R}^3)$. In particular, $W_0(x)$ is not in the space $\dot{H}^{sp}(\mathbb{R}^3)$. Actually we have $\limsup_{x \rightarrow 0^+} |x|^\theta |W_0(x)| > 0$ by the argument above. This gives us a singularity at zero.

9.3 $W_0(x)$ is smooth in $\mathbb{R}^3 \setminus \{0\}$

In this subsection, we will discover some additional properties of the soliton $W_0(x)$. Assume that $y(r)$ and $v(r)$ are defined in the same manner as the previous subsection.

$W_0(x)$ is a positive solution If this was not true, we could assume that $v(r_0) = 0$ for some $r_0 > 0$. Then by (59), we obtain

$$r^2 \frac{|v'(r)|^2}{2} - \frac{p-3}{(p-1)^2} v^2(r) + \frac{|v(r)|^{p+1}}{p+1} \geq r_0^2 \frac{|v'(r_0)|^2}{2} > 0. \quad (60)$$

for each $r > r_0$. However the decay of $W_0(x)$ implies (if r is large)

$$v(r) \lesssim r^{\theta-1};$$

$$v'(r) = \theta r^{\theta-1} y(r) + r^\theta y'(r) \lesssim r^{\theta-2}.$$

This gives us a contradiction if we consider the limit of the left hand in the inequality (60) using these estimates.

Remark Due to the fact that the function F is smooth in \mathbb{R}^+ , A direct corollary follows that the function $W_0(x)$ is smooth everywhere except for $x = 0$.

10 Appendix

10.1 The Duhamel Formula

Lemma 10.1. *Let $1/2 < s \leq 1$. If K is a compact subset of $\dot{H}^s \times \dot{H}^{s-1}$ with an s -admissible pair (q, r) so that $q \neq \infty$, then for each $\varepsilon > 0$, there exist two constants $M, \delta > 0$ such that*

$$\begin{aligned} \|S(t)(u_0, u_1)\|_{L^q L^r(J \times \mathbb{R}^3)} &+ \|S(t)(u_0, u_1)\|_{L^q L^r([M, \infty) \times \mathbb{R}^3)} \\ &+ \|S(t)(u_0, u_1)\|_{L^q L^r((-\infty, M] \times \mathbb{R}^3)} < \varepsilon \end{aligned}$$

holds for any $(u_0, u_1) \in K$ and any time interval J with a length $|J| \leq \delta$.

Proof Given $(u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-1}$, it is clear that we are able to find $M, \delta > 0$ so that the inequality holds for this particular pair of initial data and any interval J with a length $|J| \leq \delta$ by the fact $q < \infty$ and the Strichartz estimate

$$\|S(t)(u_0, u_1)\|_{L^q L^r(\mathbb{R} \times \mathbb{R}^3)} < \infty.$$

If K is a finite set, then we can find M and δ so that they work for each pair in K by taking a maximum among M 's and a minimum among δ 's. In the general case, we can just choose a finite subset $\{(u_{0,i}, u_{1,i})\}_{i=1,2,\dots,n}$ of K such that for each $(u_0, u_1) \in K$, there exists $1 \leq i \leq n$ with

$$\|S(t)(u_0 - u_{0,i}, u_1 - u_{1,i})\|_{L^q L^r(\mathbb{R} \times \mathbb{R}^3)} \leq C \| (u_0 - u_{0,i}, u_1 - u_{1,i}) \|_{\dot{H}^s \times \dot{H}^{s-1}} < 0.01\varepsilon,$$

and then use our result for a finite subset.

Lemma 10.2. (the Duhamel formula) *Let $u(x, t)$ be almost periodic modulo scaling in the interval $I = (T_-, \infty)$, namely the set*

$$K = \left\{ \left(\frac{1}{\lambda(t)^{3/2-s_p}} u \left(\frac{x}{\lambda(t)}, t \right), \frac{1}{\lambda(t)^{5/2-s_p}} \partial_t u \left(\frac{x}{\lambda(t)}, t \right) \right) : t \in I \right\}$$

is precompact in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$. Then for any time $t_0 \in \mathbb{R}$, any bounded closed interval $[a, b]$ and an s_p -admissible pair (q, r) with $q < \infty$, we have

$$\lim_{T \rightarrow +\infty} \|S(t-T)(u(T), \partial_t u(T))\|_{L^q L^r([a, b] \times \mathbb{R}^3)} = 0.$$

$$\text{weak} \lim_{T \rightarrow +\infty} S(t_0 - T) \begin{pmatrix} u(T) \\ \partial_t u(T) \end{pmatrix} = 0.$$

Proof We have

$$\begin{aligned}
& \|S(t-T)(u(T), \partial_t u(T))\|_{L^q L^r([a,b] \times \mathbb{R}^3)} \\
&= \|S(t)(u(T), \partial_t u(T))\|_{L^q L^r([a-T, b-T] \times \mathbb{R}^3)} \\
&= \left\| S(t)(u_0^{(T)}, u_1^{(T)}) \right\|_{L^q L^r([\lambda(T)(a-T), \lambda(T)(b-T)] \times \mathbb{R}^3)},
\end{aligned}$$

here

$$(u_0^{(T)}, u_1^{(T)}) = \left(\frac{1}{\lambda(T)^{3/2-s_p}} u \left(\frac{\cdot}{\lambda(T)}, T \right), \frac{1}{\lambda(T)^{5/2-s_p}} \partial_t u \left(\frac{\cdot}{\lambda(T)}, T \right) \right).$$

Given $\varepsilon > 0$, let M, δ be the constants as in lemma 10.1. It is clear that if T is sufficiently large, we have either $(\lambda(T) \text{ is small})$

$$\lambda(T)(b-T) - \lambda(T)(a-T) = (b-a)\lambda(T) < \delta;$$

or $(\lambda(T) \text{ is large})$

$$\lambda(T)(b-T) < -M.$$

In either case, by lemma 10.1 we have $\|S(t-T)(u(T), \partial_t u(T))\|_{L^q L^r([a,b] \times \mathbb{R}^3)} < \varepsilon$. This completes the proof of the first limit.

In order to obtain the second limit, we only need to choose $t_1 \in (t_0, +\infty)$, set $[a, b] = [t_0, t_1]$ and apply lemma 10.3 below using the first limit and the following identity.

$$S(t-t_0) \left[S(t_0-T) \begin{pmatrix} u(T) \\ \partial_t u(T) \end{pmatrix} \right] = S(t-T) \begin{pmatrix} u(T) \\ \partial_t u(T) \end{pmatrix}.$$

Remark We can obtain the similar result in the negative time direction using exactly the same argument. This implies the corresponding Duhamel formula in the negative time direction.

- Soliton-like Case or High-to-low Frequency Cascade Case

$$\lim_{T \rightarrow -\infty} \|S(t-T)(u(T), \partial_t u(T))\|_{L^q L^r([a,b] \times \mathbb{R}^3)} = 0.$$

$$\text{weak } \lim_{T \rightarrow -\infty} S(t_0-T) \begin{pmatrix} u(T) \\ \partial_t u(T) \end{pmatrix} = 0.$$

- Self-similar Case (let $a, t_0 > 0$)

$$\lim_{T \rightarrow 0^+} \|S(t-T)(u(T), \partial_t u(T))\|_{L^q L^r([a,b] \times \mathbb{R}^3)} = 0.$$

$$\text{weak } \lim_{T \rightarrow 0^+} S(t_0-T) \begin{pmatrix} u(T) \\ \partial_t u(T) \end{pmatrix} = 0.$$

Lemma 10.3. Suppose that $\{(u_{0,n}, u_{1,n})\}_{n \in \mathbb{Z}}$ is a bounded subset of $\dot{H}^s \times \dot{H}^{s-1}$ so that

$$\lim_{n \rightarrow \infty} \|S(t)(u_{0,n}, u_{1,n})\|_{L^q L^r([0, \mu] \times \mathbb{R}^3)} = 0.$$

Here (q, r) is an s -admissible pair and μ is a positive constant. Then we have the following weak limit in $\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^3)$,

$$(u_{0,n}, u_{1,n}) \rightharpoonup 0.$$

Proof Let us suppose the conclusion was false. This means that there exists a subsequence (WLOG, let us use the same notation as the original sequence) so that it converges weakly to a nonzero limit $(\tilde{u}_0, \tilde{u}_1)$. We know the operator $P : \dot{H}^s \times \dot{H}^{s-1} \rightarrow L^q L^r([0, \mu] \times \mathbb{R}^3)$ defined by

$$P(u_0, u_1) = S(t)(u_0, u_1)$$

is bounded by the Strichartz estimate. This implies that we have the weak limit below in $L^q L^r([0, \mu] \times \mathbb{R}^3)$

$$P(u_{0,n}, u_{1,n}) \rightharpoonup P(\tilde{u}_0, \tilde{u}_1).$$

On the other hand, we know $P(u_{0,n}, u_{1,n})$ converges to zero strongly. Thus $P(\tilde{u}_0, \tilde{u}_1) = 0$. This means $(\tilde{u}_0, \tilde{u}_1) = 0$, which is a contradiction.

Lemma 10.4. Assume $s \in [s_p, 1]$. Let $u(x, t)$ be defined on $I = (T_-, \infty)$ and almost periodic modulo scalings in $\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^3)$, namely the set

$$K = \left\{ \left(\frac{1}{\lambda(t)^{3/2-s_p}} u \left(\frac{x}{\lambda(t)}, t \right), \frac{1}{\lambda(t)^{5/2-s_p}} \partial_t u \left(\frac{x}{\lambda(t)}, t \right) \right) : t \in I \right\}$$

is precompact in the space $\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^3)$. In addition $\lambda(t) \leq 1$ for each $t \geq 1$. Then for any closed interval $[a, b] \subset I$ and any s -admissible pair (q, r) with $q < \infty$, we have

$$\lim_{T \rightarrow +\infty} \|S(t-T)(u(T), \partial_t u(T))\|_{L^q L^r([a, b] \times \mathbb{R}^3)} = 0.$$

Proof One could use the similar method as used in lemma 10.2 by observing

$$\begin{aligned} & \|S(t-T)(u(T), \partial_t u(T))\|_{L^q L^r([a, b] \times \mathbb{R}^3)} \\ &= \|S(t)(u(T), \partial_t u(T))\|_{L^q L^r([a-T, b-T] \times \mathbb{R}^3)} \\ &= (\lambda(T))^{s-s_p} \left\| S(t)(u_0^{(T)}, u_1^{(T)}) \right\|_{L^q L^r([\lambda(T)(a-T), \lambda(T)(b-T)] \times \mathbb{R}^3)}. \end{aligned}$$

Here

$$(u_0^{(T)}, u_1^{(T)}) = \left(\frac{1}{\lambda(T)^{3/2-s_p}} u \left(\frac{\cdot}{\lambda(T)}, T \right), \frac{1}{\lambda(T)^{5/2-s_p}} \partial_t u \left(\frac{\cdot}{\lambda(T)}, T \right) \right).$$

10.2 Perturbation Theory

Proof of theorem 2.6 Let us first prove the perturbation theory when M is sufficiently small. Let I_1 be the maximal lifespan of the solution $u(x, t)$ to the equation (1) with the given initial data (u_0, u_1) and assume $[0, T] \subseteq I \cap I_1$. By the Strichartz estimate, we have

$$\begin{aligned} \|\tilde{u} - u\|_{Y_{s_p}([0, T])} &\leq \|S(t)(u_0 - \tilde{u}(0), u_1 - \tilde{u}(0))\|_{Y_{s_p}([0, T])} + C_p \|e + F(\tilde{u}) - F(u)\|_{Z_{s_p}([0, T])} \\ &\leq \varepsilon + C_p \|e\|_{Z_{s_p}([0, T])} + C_p \|F(\tilde{u}) - F(u)\|_{Z_{s_p}([0, T])} \\ &\leq \varepsilon + C_p \varepsilon + C_p \|\tilde{u} - u\|_{Y_{s_p}([0, T])} (\|\tilde{u}\|_{Y_{s_p}([0, T])}^{p-1} + \|\tilde{u} - u\|_{Y_{s_p}([0, T])}^{p-1}) \\ &\leq C_p \varepsilon + C_p \|\tilde{u} - u\|_{Y_{s_p}([0, T])} (M^{p-1} + \|\tilde{u} - u\|_{Y_{s_p}([0, T])}^{p-1}). \end{aligned}$$

By a continuity argument in T , there exist $M_0 = M_0(p), \varepsilon_0 = \varepsilon_0(p) > 0$, such that if $M \leq M_0$ and $\varepsilon < \varepsilon_0$, we have

$$\|\tilde{u} - u\|_{Y_{s_p}([0, T])} \leq C_p \varepsilon.$$

Observing that this estimate does not depend on the time T , we are actually able to conclude $I \subseteq I_1$ by the standard blow-up criterion and obtain

$$\|\tilde{u} - u\|_{Y_{sp}(I)} \leq C_p \varepsilon.$$

In addition, by the Strichartz estimate

$$\begin{aligned} & \sup_{t \in I} \left\| \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} - \begin{pmatrix} \tilde{u}(t) \\ \partial_t \tilde{u}(t) \end{pmatrix} - S(t) \begin{pmatrix} u_0 - \tilde{u}(0) \\ u_1 - \partial_t \tilde{u}(0) \end{pmatrix} \right\|_{\dot{H}^{sp} \times \dot{H}^{sp-1}} \\ & \leq C_p \|F(u) - F(\tilde{u}) - e\|_{Z_{sp}(I)} \\ & \leq C_p \left(\|e\|_{Z_{sp}(I)} + \|F(u) - F(\tilde{u})\|_{Z_{sp}(I)} \right) \\ & \leq C_p \left[\varepsilon + \|u - \tilde{u}\|_{Y_{sp}(I)} \left(\|\tilde{u}\|_{Y_{sp}(I)}^{p-1} + \|u - \tilde{u}\|_{Y_{sp}(I)}^{p-1} \right) \right] \\ & \leq C_p \varepsilon. \end{aligned}$$

This finishes the proof as M is sufficiently small. To deal with the general case, we can separate the time interval I into finite number of subintervals $\{I_j\}$, so that $\|\tilde{u}\|_{Y_{sp}(I_j)} < M_0$, and then iterate our argument above.

Proof of theorem 2.8 Let us first prove the perturbation theory when M and T are sufficiently small. Let I_1 be the maximal lifespan of the solution $u(x, t)$ to the equation (1) with the given initial data (u_0, u_1) and assume $[0, T_1] \subseteq [0, T] \cap I_1$. By the Strichartz estimate, we have

$$\begin{aligned} \|\tilde{u} - u\|_{Y_s([0, T_1])} & \leq \|S(t)(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{Y_s([0, T_1])} + C_{s,p} \|F(\tilde{u}) - F(u)\|_{Z_s([0, T_1])} \\ & \leq C_{s,p} \|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} + C_{s,p} \|F(\tilde{u}) - F(u)\|_{Z_s([0, T_1])} \\ & \leq C_{s,p} \|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} \\ & \quad + C_{s,p} T_1^{(p-1)(s-sp)} \|F(\tilde{u}) - F(u)\|_{L^{\frac{2}{s+1-(2p-2)(s-sp)}} L^{\frac{2}{2-s}}([0, T_1] \times \mathbb{R}^3)} \\ & \leq C_{s,p} \|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} \\ & \quad + C_{s,p} T_1^{(p-1)(s-sp)} \|\tilde{u} - u\|_{Y_s([0, T_1])} \left(\|\tilde{u} - u\|_{Y_s([0, T_1])}^{p-1} + \|\tilde{u}\|_{Y_s([0, T_1])}^{p-1} \right) \\ & \leq C_{s,p} \|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} \\ & \quad + C_{s,p} T_1^{(p-1)(s-sp)} \|\tilde{u} - u\|_{Y_s([0, T_1])} \left(\|\tilde{u} - u\|_{Y_s([0, T_1])}^{p-1} + M^{p-1} \right). \end{aligned}$$

By a continuity argument in T_1 , there exist $M_0 = M_0(s, p)$, $\varepsilon_0 = \varepsilon_0(s, p) > 0$, such that if $M \leq M_0$, $T \leq 1$ and $\|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} \leq \varepsilon_0$, we have

$$\|\tilde{u} - u\|_{Y_s([0, T_1])} \leq C_{s,p} \|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}}.$$

Observing that this estimate does not depend on the time T_1 as long as $T_1 \leq T \leq 1$, we are actually able to conclude $[0, T] \subseteq I_1$ by theorem 2.7 and obtain

$$\|\tilde{u} - u\|_{Y_s([0, T])} \leq C_{s,p} \|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}}.$$

In addition, by the Strichartz estimate

$$\begin{aligned}
& \sup_{t \in I} \left\| \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} - \begin{pmatrix} \tilde{u}(t) \\ \partial_t \tilde{u}(t) \end{pmatrix} \right\|_{\dot{H}^s \times \dot{H}^{s-1}} \\
& \leq \left\| S(t) \begin{pmatrix} u_0 - \tilde{u}_0 \\ u_1 - \tilde{u}_1 \end{pmatrix} \right\|_{\dot{H}^s \times \dot{H}^{s-1}} + C_{s,p} \|F(u) - F(\tilde{u})\|_{Z_s([0,T])} \\
& \leq C_{s,p} \|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} \\
& \quad + C_{s,p} T^{(p-1)(s-s_p)} \|\tilde{u} - u\|_{Y_s([0,T])} (\|\tilde{u} - u\|_{Y_s([0,T])}^{p-1} + \|\tilde{u}\|_{Y_s([0,T])}^{p-1}) \\
& \leq C_{s,p} \|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}}.
\end{aligned}$$

This finishes the proof as M and T are sufficiently small. To deal with the general case, we can separate the time interval $[0, T]$ into finite number of subintervals $\{I_j\}$, so that $\|\tilde{u}\|_{Y_s(I_j)} \leq M_0$ and $|I_j| \leq 1$, then iterate our argument above.

10.3 Technical Lemmas

Lemma 10.5. *Suppose that $(u_{0,\varepsilon}(x), u_{1,\varepsilon}(x))$ are radial, smooth pairs defined in \mathbb{R}^3 and converge to $(u_0(x), u_1(x))$ strongly in $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$. In addition, we have*

$$\int_{r_0 < |x| < 4r_0} (|\nabla u_{0,\varepsilon}(x, t_0)|^2 + |u_{1,\varepsilon}(x, t_0)|^2) dx \leq C$$

for each $\varepsilon < \varepsilon_0$. Then $(u_0(x), u_1(x))$ is in the space $\dot{H}^1 \times L^2(r < |x| < 4r)$ and satisfies

$$\int_{r_0 < |x| < 4r_0} (|\nabla u_0(x)|^2 + |u_1(x)|^2) dx \leq C.$$

Proof By the uniform bound of the integral, we can extract a sequence $\varepsilon_i \rightarrow 0$ so that $\partial_r u_{0,\varepsilon_i}(r)$ converges to $\tilde{u}'_0(r)$ weakly in $L^2(r_0, 4r_0)$, and u_{1,ε_i} converges to \tilde{u}_1 weakly in $L^2(r_0 < |x| < 4r_0)$. Define

$$\tilde{u}_0(r) = u_0(r_0) + \int_{r_0}^r \tilde{u}'_0(\tau) d\tau.$$

We have

$$\int_{r_0 < |x| < 4r_0} (|\nabla \tilde{u}_0(x)|^2 + |\tilde{u}_1(x)|^2) dx \leq C.$$

By the strong and weak convergence, we have immediately $u_1 = \tilde{u}_1$ in the region $r_0 < |x| < 4r_0$. In order to conclude, we only need to show $u_0(r) = \tilde{u}_0(r)$. Observing $\int_{r_0}^{r_1} f(\tau) d\tau$ is a bounded linear functional in $L^2(r_0, 4r_0)$ for each $r_1 \in (r_0, 4r_0)$, we have

$$\begin{aligned}
\tilde{u}_0(r_1) &= u_0(r_0) + \int_{r_0}^{r_1} \tilde{u}'_0(\tau) d\tau \\
&= \lim_{i \rightarrow \infty} u_{0,\varepsilon_i}(r_0) + \lim_{i \rightarrow \infty} \int_{r_0}^{r_1} \partial_r u_{0,\varepsilon_i}(\tau) d\tau \\
&= \lim_{i \rightarrow \infty} \left(u_{0,\varepsilon_i}(r_0) + \int_{r_0}^{r_1} \partial_r u_{0,\varepsilon_i}(\tau) d\tau \right) \\
&= \lim_{i \rightarrow \infty} u_{0,\varepsilon_i}(r_1) \\
&= u_0(r_1).
\end{aligned}$$

This completes the proof.

Lemma 10.6. *There exists a constant $\kappa = \kappa(p) \in (0, 1)$ that depends only on p , so that for each $s \in [s_p, 1)$, there exists an s -admissible pair (q, r) , with $q \neq \infty$ and*

$$\frac{s+1-(2p-2)(s-s_p)}{2p} = \kappa \cdot 0 + (1-\kappa)\frac{1}{q}; \quad \frac{2-s}{2p} = \kappa\frac{3-2s}{6} + (1-\kappa)\frac{1}{r}.$$

Proof We will choose $\kappa = 1 - \frac{3}{p} \in (0, 0.4)$. Basic Computation shows

$$\frac{1}{q} = \frac{s+1-(2p-2)(s-s_p)}{2p(1-\kappa)} = \frac{s+1-(2p-2)(s-s_p)}{6} \in (0, 1/3);$$

$$\begin{aligned} \frac{1}{r} &= \frac{2-s}{2p(1-\kappa)} - \frac{\kappa}{1-\kappa} \times \frac{3-2s}{6} \\ &= \frac{2-s}{6} - \frac{\kappa}{1-\kappa} \times \frac{3-2s}{6} \\ &\in \left(\frac{2-s}{6} - \frac{2}{3} \times \frac{3-2s}{6}, \frac{2-s}{6} \right) \\ &\subseteq \left(\frac{s}{18}, \frac{2-s}{6} \right) \\ &\subseteq (1/36, 1/4) \end{aligned}$$

Thus we can solve two positive real number q, r so that the two identities hold. In addition, we have $q \in (3, \infty)$ and $r \in (4, 36)$. Furthermore, by adding the identities together, we obtain

$$\frac{3-(2p-2)(s-s_p)}{2p} = \kappa\frac{3-2s}{6} + (1-\kappa)\left(\frac{1}{q} + \frac{1}{r}\right)$$

This implies

$$\frac{1}{q} + \frac{1}{r} < \frac{3-(2p-2)(s-s_p)}{2p(1-\kappa)} = \frac{3-(2p-2)(s-s_p)}{6} \leq 1/2.$$

Using the same method, one can show $1/q + 3/r = 3/2 - s$. In summary, (q, r) is an s -admissible pair.

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